NONLINEAR RECURSIVE FILTER FOR BOOST TRAJECTORIES

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Abstract. A nonlinear recursive algorithm is formulated for state vector and covariance estimation of boost trajectories. The thrust acceleration vector of the booster is modeled by a vector-differential equation that includes effects of propellant depletion and attitude motions resulting from gravity-turn maneuvers and other steering maneuvers. This new dynamics model is incorporated in an extended Kalman filter with nine state variables that describe the inertial components of position, velocity and thrust acceleration. Additional algorithms are described for filter initialization using angle-only measurements from a geostationary sensor and for detection and estimation of the final staging event using measurement residuals. Tracking accuracy and covariance fidelity are assessed by Monte-Carlo simulation.

I. Introduction

Many algorithms have been developed for tracking a maneuvering vehicle.[1-6] An important concern is filter stability when target maneuvers are inaccurately or incompletely modeled, and an adaptive filtering approach is often suggested.[7-10] Alternatively, measurements can be processed concurrently with several models of expected maneuvers, and criteria can be developed for switching among models based on statistical hypothesis tests.[11-13]

Ballistic missiles perform a somewhat predictable sequence of maneuvers from vertical launch at the Earth's surface to orbital insertion. As the missile accelerates through the atmosphere, gravitational acceleration rotates the flight path downward while the booster maintains small angles of attack, lateral aerodynamic forces and de-stabilizing aerodynamic torques. Outside the atmosphere, guidance steering algorithms may command large changes in booster orientation and flight path. When formulated using a reference boost trajectory[14] (or *template*), boost filters often diverge because the observed lofting profile deviates from the template.

In this article, a nonlinear recursive algorithm is formulated for state vector and covariance estimation of boost trajectories. This estimation algorithm does not need a booster template because the dynamical effects of propellant depletion and vehicle angular dynamics are modeled with a new vector-differential equation for the thrust acceleration vector of the booster. As the booster burns propellant and loses mass, thrust-acceleration magnitude increases at a rate proportional to the square of the thrust acceleration. The thrust vector is parallel to the longitudinal axis of the booster, which rotates during gravity-turn maneuvers and other steering maneuvers. This new model is incorporated in an extended Kalman filter with nine state variables that describe the inertial components of position, velocity and thrust acceleration. This filter is an extension of earlier research on improved algorithms for tracking and interception of accelerated targets during reentry[15] and boost.[16]

Because of the complexity of the filter dynamics model, the tracking algorithm requires accurate, three-dimensional measurements of position at frequent intervals. Three-dimensional position may be measured using a ground-based radar, or with a suite of (at least) two angle-only sensors. The latter case is examined in detail to quantify filter performance sensitivity to measurement accuracy and availability (or update rate).

Filter initialization can be problematic when boost sensor(s) cannot observe the launch event directly. In this situation, an approximate state estimate *at launch* may be synthesized from observations after launch. This state estimate and an approximate *a-priori* covariance matrix are

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propagated from launch to the first sensor measurement using the nonlinear dynamical model. This technique for filter initialization establishes the proper correlation statistics among the position, velocity and acceleration errors. A properly-correlated initial covariance matrix can enhance the effectiveness of the first measurement update and can accelerate convergence of the track filter.[17]

When observations of the post-boost trajectory are available, the final thrust-termination event may be detected because the filter measurement residuals diverge. After detection and estimation of the burnout time, the post-boost measurements are re-processed with a thrust-decay model in order to improve accuracy of the burnout state and covariance estimates.

The new filter dynamics model is formulated (Section II) and incorporated in a nonlinear recursive filter (Section III). Additional algorithms are described for filter initialization (Section IV) and for detection and estimation of the final staging event (Section V). Performance of the nonlinear tracking filter will be demonstrated by Monte-Carlo simulation (Section VI). Effects of clutter and data association on track accuracy are not considered herein. The significance of this work is summarized and areas for further research are suggested (Section VII).

II. Filter Dynamics Model

A nonlinear dynamical model is formulated in Earth-Centered Inertial (or ECI) coordinates. Booster translation is described by a three degree-of-freedom system:

$$\frac{d\vec{v}}{dt} = \vec{a}_c + \vec{a}_a + \vec{g}, \qquad \frac{d\vec{r}}{dt} = \vec{v}$$

where $\vec{r}$ is the ECI position vector, $\vec{v}$ is the inertial velocity vector, and time derivatives are taken with respect to an observer in the ECI frame. The resultant acceleration consists of three terms. Control acceleration $\vec{a}_c$ arises from forces for translational control and attitude control. Aerodynamic acceleration $\vec{a}_a(\vec{r},\vec{v})$ includes lift and drag accelerations during atmospheric flight.

Gravitational acceleration $\vec{g}(\vec{r})$ is approximated by the central, inverse-square gravitational field. Although small velocity errors arise from the omission of gravitational perturbations from a non-spherical field, these errors will be dominated by other measurement-related errors.

The six translational equations must be augmented with a vector-differential equation for $\vec{a}_c$. During boost, the dominant source of control acceleration arises from thrust acceleration, which is parallel to the unit longitudinal axis $\vec{e}$ of the vehicle:

$$\vec{a}_c = a_c \vec{e}$$

Differentiation of this identity generates a vector-differential equation:

$$\frac{d\vec{a}_c}{dt} = \dot{a}_c \vec{e} + \vec{\omega} \times \vec{a}_c$$

where $\vec{e}$ rotates at angular vector $\vec{\omega}$ relative to inertial space. When propellant massflow rate and exhaust velocity $U(= I_{sp} g)$ are constants, the magnitude variation of thrust acceleration may be approximated by (refer to Appendix A):

$$\dot{a}_c = \frac{a_c^2}{U} \qquad \left(\ddot{m} = 0, \dot{U} = 0\right)$$

Using this identity and the preceding definition of $\vec{a}_c$, it may be shown that $\vec{e}$ may be eliminated from the term $\dot{a}_c \vec{e}$, and the vector-differential equation takes the form:

$$\frac{d\vec{a}_c}{dt} = \left(\frac{a_c}{U}\right)\vec{a}_c + \vec{\omega} \times \vec{a}_c$$

The Coriolis term $\vec{\omega} \times \vec{a}_c$ describes the directional change of $\vec{a}_c$ caused by attitude motions.

A quasi-equilibrium model for $\vec{\omega}$ may be derived when $\vec{e}$ maintains a fixed orientation (or angle of attack) with respect to the velocity $\vec{u}(= \vec{v} - \vec{\omega}_e \times \vec{r})$ relative to a wind-free atmosphere that co-rotates with the Earth. During an accelerated turn at constant angle of attack, there are two sources of angular velocity (refer to Appendix B):

$$\vec{\omega} = \vec{\omega}_g + \vec{\omega}_c$$

$$\vec{\omega}_g = \frac{\mu_e}{r^3 u^2}\left(\vec{r} \times \vec{u}\right), \qquad \vec{\omega}_c = \frac{1}{u^2}\left(\vec{u} \times \vec{a}_c\right)$$

During a *gravity-turn* maneuver at small angle of attack, $\vec{\omega}_g$ arises from the downward rotation of $\vec{u}$ (and hence $\vec{e}$) caused by gravity. During an accelerated turn when the booster is controlled at non-zero angles of attack, $\vec{\omega}_c$ arises from the rotation of $\vec{u}$ in the direction of the non-zero component of $\vec{a}_c$ perpendicular to $\vec{u}$.

Attitude motions of the booster are modeled implicitly in the equations of motion. For example, translational dynamics are correlated with attitude dynamics through $\vec{\omega}\left(\vec{r}, \vec{u}, \vec{a}_c\right)$. Moreover, initial conditions on pitch-yaw orientation are implicit in $\vec{a}_c(0)$, and a time history of pitch-yaw attitude may be inferred from $\vec{a}_c(t)$ because it is parallel to the longitudinal axis. Although roll orientation cannot be determined from $\vec{a}_c$ alone, changes in roll orientation generally do not generate significant thrust or aerodynamic forces during boost.

These filter models are minimal representations for $\vec{a}_c$ and $\vec{\omega}$ because additional state variables and differential equations are not needed. Consequently, it is not necessary to model the detailed dynamics of the thrust motor (e.g., valve, nozzle flow, and combustion processes), and the booster guidance and control algorithms. The latter consideration is important because these algorithms are usually unknown and cannot be included in the filter model. A representative value of exhaust velocity U essentially identifies the booster.

Ordinarily, additional state variables and differential equations would be needed to describe $\vec{a}_a$. At small angles of attack, the drag acceleration vector may be described by a linear dynamical model for limited time intervals.[15] Alternatively, the filter could be reformulated to estimate the difference of thrust minus drag.[18] However, other complications arise at non-zero angles of attack because lift acceleration (perpendicular to $\vec{u}$) would be generated. In our formulation, aerodynamic drag is neglected compared to thrust because the booster is massive, and lift acceleration is neglected because angles of attack are usually small during atmospheric flight.

III. Nonlinear Recursive Filter

Several techniques for nonlinear estimation include minimum-variance estimation, estimation by statistical linearization, and batch least-squares estimation.[1-2] The minimum-variance technique is implemented because of its simplicity, from a computational standpoint, compared to the other two methods. The simplest form of the minimum-variance estimator is the extended Kalman filter (EKF). The EKF consists of mathematical equations for the filter update, and for state and covariance propagation, as follows.

For accelerated motion in a gravitational field, a minimal representation for the state vector $\vec{x}$ includes the position vector $\vec{r}$, inertial velocity vector $\vec{v}$ and thrust acceleration vector $\vec{a}_c$:

$$\vec{x}(t) = \begin{bmatrix} \vec{r}(t) \\ \vec{v}(t) \\ \vec{a}_c(t) \end{bmatrix}$$

As in earlier studies,[15,16] three state variables describe the ECI components of $\vec{a}_c$, which is dominated by the vehicle thrust. The measurement $\vec{y}$ and the observation matrix H are nonlinear functions of $\vec{x}$:

$$\vec{y} = \vec{h}(\vec{x}) + \vec{\upsilon}, \quad H(\vec{x}) = \frac{\partial \vec{h}}{\partial \vec{x}^T}$$

Measurement errors $\vec{\upsilon}$ are approximated by an additive, zero-mean Gaussian noise process:

$$E\{\vec{\upsilon}\} = 0, \quad R = E\{\vec{\upsilon}\vec{\upsilon}^T\}$$

where the measurement-noise matrix R is a diagonal matrix in sensor coordinates.

After filter initialization occurs, the EKF state estimate $\hat{\vec{x}}_n$ and covariance matrix P_n are updated sequentially when a measurement $\vec{y}_n$ becomes available at time t_n:

$$K_n = M_n H_n^T \left(H_n M_n H_n^T + R_n\right)^{-1}$$

$$P_n = M_n - K_n H_n M_n$$

$$\hat{\vec{x}}_n = \bar{\vec{x}}_n + K_n \vec{\varepsilon}_n, \quad \vec{\varepsilon}_n = \vec{y}_n - \vec{h}\left(\bar{\vec{x}}_n\right)$$

The prior state estimate $\bar{\vec{x}}_n$ is corrected with a gain-weighted residual $\vec{\varepsilon}_n$. As $\vec{y}_n$ and R_n are specified in sensor coordinates, the filter gain matrix K_n depends on $\bar{\vec{x}}_n$ because $H_n = H\left(\bar{\vec{x}}_n\right)$.

An important limitation of the EKF is that the prior covariance M_n is not updated using the *actual statistics* of $\vec{\varepsilon}_n$. Instead, these statistics are inferred from the identity:

$$E\left\{\vec{\varepsilon}_n \vec{\varepsilon}_n^T\right\} = N_n, \quad N_n = H_n M_n H_n^T + R_n$$

Although this identity is valid for a linear process, nonlinear effects can cause differences in between the actual and filter-modeled statistics such that $E\left\{\vec{\varepsilon}_n \vec{\varepsilon}_n^T\right\} \neq N_n$. Consequently, P_n may not accurately represent the statistics of actual errors in the estimates. This *covariance fidelity* issue is addressed in the Monte-Carlo performance evaluation.

Using $\hat{\vec{x}}_n$ and P_n as initial conditions, $\bar{\vec{x}}_{n+1}$ and M_{n+1} are predicted at the time t_{n+1} of the next observation by integration of a system of nonlinear differential equations (refer to Table 1). Nine differential equations for $\hat{\vec{x}}_n$ are coupled to the 45 differential equations for P_n because the linearized dynamics matrix $F(\vec{x}) = \dfrac{\partial \vec{f}}{\partial \vec{x}^T}$ is a nonlinear function of the state. Because of this nonlinearity, errors in $\hat{\vec{x}}_n$ will cause errors in $F\left(\hat{\vec{x}}_n\right)$, and process noise Q is needed because these errors are not properly compensated in the equations for covariance propagation.[17] As an analytic model for Q is not derived in this work, a diagonal Q matrix is assigned with constant variance q_a (to be discussed later).

The elements of $F(\vec{x})$ may be determined by partial differentiations of the nonlinear filter equations, using the following identities (refer to Table 1):

$$A = \frac{\partial \vec{\alpha}}{\partial \vec{r}^T}, \quad B = \frac{\partial \vec{\alpha}}{\partial \vec{v}^T}, \quad C = \frac{\partial \vec{\alpha}}{\partial \vec{a}_c^T}$$

$$\frac{\partial \vec{\alpha}}{\partial r_i} = \frac{\partial \vec{\omega}}{\partial r_i} \times \vec{a}_c \quad (i = 1,2,3)$$

$$\frac{\partial \vec{\alpha}}{\partial v_i} = \frac{\partial \vec{\omega}}{\partial v_i} \times \vec{a}_c \quad (i = 1,2,3)$$

$$\frac{\partial \vec{\alpha}}{\partial a_i} = \frac{a_c}{U}\frac{\partial \vec{a}_c}{\partial a_i} + \frac{1}{U}\frac{\partial a_c}{\partial a_i}\vec{a}_c + \vec{\omega} \times \frac{\partial \vec{a}_c}{\partial a_i} + \frac{\partial \vec{\omega}}{\partial a_i} \times \vec{a}_c \quad (i = 1,2,3)$$

where U is the exhaust velocity, and r_i, v_i, a_i are the components of $\vec{r}, \vec{v}, \vec{a}_c$, respectively. After some algebra, it may be shown that A, B, C may be expressed by:

$$A = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(r_1, r_2, r_3)}$$

$$B = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(v_1, v_2, v_3)}$$

$$C = \begin{bmatrix} \dfrac{a_c^2 + a_1^2}{Ua_c} & \dfrac{a_1 a_2}{Ua_c} - \omega_3 & \dfrac{a_1 a_3}{Ua_c} + \omega_2 \\ \dfrac{a_1 a_2}{Ua_c} + \omega_3 & \dfrac{a_c^2 + a_2^2}{Ua_c} & \dfrac{a_2 a_3}{Ua_c} - \omega_1 \\ \dfrac{a_1 a_3}{Ua_c} - \omega_2 & \dfrac{a_2 a_3}{Ua_c} + \omega_1 & \dfrac{a_c^2 + a_3^2}{Ua_c} \end{bmatrix} + \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(a_1, a_2, a_3)}$$

where ω_i denotes a component of $\vec{\omega}$. Three Jacobian matrices contain partial derivatives of $\vec{\omega}$ with respect to $\vec{r}, \vec{v}, \vec{a}_c$. Partial derivatives with respect to the position, velocity and thrust-acceleration components are given by:

$$\frac{\partial \vec{\omega}}{\partial r_i} = \frac{\vec{u}}{u^2} \times \frac{\partial \vec{g}}{\partial r_i} + \frac{\partial}{\partial r_i}\left(\frac{\vec{u}}{u^2}\right) \times \left(\vec{g} + \vec{a}_c\right) \quad (i = 1,2)$$

$$\frac{\partial \vec{\omega}}{\partial r_3} = \frac{\vec{u}}{u^2} \times \frac{\partial \vec{g}}{\partial r_3}$$

$$\frac{\partial}{\partial r_1}\left(\frac{\vec{u}}{u^2}\right) = \frac{\omega_e}{u^4}\begin{bmatrix} -2u_1u_2 \\ u^2 - 2u_2^2 \\ -2u_2u_3 \end{bmatrix}$$

$$\frac{\partial}{\partial r_2}\left(\frac{\vec{u}}{u^2}\right) = -\frac{\omega_e}{u^4}\begin{bmatrix} u^2 - 2u_1^2 \\ -2u_1u_2 \\ -2u_1u_3 \end{bmatrix}$$

$$\frac{\partial \vec{\omega}}{\partial v_i} = \frac{\partial}{\partial v_i}\left(\frac{\vec{u}}{u^2}\right) \times \left(\vec{g} + \vec{a}_c\right) \quad (i = 1,2,3)$$

$$\frac{\partial}{\partial \vec{v}^T}\left(\frac{\vec{u}}{u^2}\right) = \frac{1}{u^4}\left(u^2 I_3 - 2\vec{u}\vec{u}^T\right)$$

$$\frac{\partial \vec{\omega}}{\partial a_i} = \left(\frac{\vec{u}}{u^2}\right) \times \frac{\partial \vec{a}_c}{\partial a_i} \quad (i = 1,2,3)$$

$$\frac{\partial \vec{a}_c}{\partial a_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{a}_c}{\partial a_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{\partial \vec{a}_c}{\partial a_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Table 1. Dynamical Model for Boost Trajectory EKF

Differential Equations	Definition of Terms
$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) = \begin{bmatrix} \vec{v} \\ \vec{a}_c + \vec{g}(\vec{r}) \\ \vec{\alpha}(\vec{r},\vec{v},\vec{a}_c) \end{bmatrix}$	$\vec{\alpha} = \frac{a_c}{U}\vec{a}_c + \vec{\omega}\times\vec{a}_c, \ \ \vec{\omega} = \frac{1}{u^2}\vec{u}\times(\vec{g}+\vec{a}_c)$ $\vec{u} = \vec{v} - \vec{\omega}_e \times \vec{r}, \ \ \vec{g} = -\frac{\mu_e}{r^3}\vec{r}, \ \ U = I_{sp} g$
$\frac{dP}{dt} = F(\vec{x})P + PF(\vec{x})^T + Q$	$F(\vec{x}) = \frac{\partial \vec{f}}{\partial \vec{x}^T} = \begin{bmatrix} O_3 & I_3 & O_3 \\ \Gamma & O_3 & I_3 \\ A & B & C \end{bmatrix}$ $\Gamma = \frac{\partial \vec{g}}{\partial \vec{r}^T} = \frac{\mu_e}{r^5}\left(3\vec{r}\vec{r}^T - r^2 I_3\right)$ $A = \frac{\partial \vec{\alpha}}{\partial \vec{r}^T}, \ \ B = \frac{\partial \vec{\alpha}}{\partial \vec{v}^T}, \ \ C = \frac{\partial \vec{\alpha}}{\partial \vec{a}_c^T}$ $Q = \begin{bmatrix} O_3 & O_3 & O_3 \\ O_3 & O_3 & O_3 \\ O_3 & O_3 & Q_a \end{bmatrix}, \ \ Q_a = \begin{bmatrix} q_a & 0 & 0 \\ 0 & q_a & 0 \\ 0 & 0 & q_a \end{bmatrix}$

IV. Filter Initialization Using an Angle-Only Sensor

The recursive filter is initialized at the time t_0 of the first observation, which is not necessarily at launch. Initial conditions $\bar{\vec{x}}(t_0)$ and $M(t_0)$ may be determined from an approximate state estimate $\bar{\vec{x}}_L$ and covariance matrix M_L *at launch*, as follows. This technique establishes the proper correlation statistics among the error states at t_0, which enhances the effectiveness of the first update.[17]

For an angle-only sensor in geostationary orbit, the first observation occurs when the radiant intensity of the booster plume (after atmospheric attenuation) exceeds the detection sensitivity of the sensor. On cloudy days, the first observation occurs above the cloud tops. As the booster is launched from the Earth's surface, the sensor does not observe the launch event.

An approximate launch position $\vec{r}_L$ may be determined from the known location $\vec{R}_s(t_0)$ of the geostationary sensor, the measured line-of-sight (LOS) unit vector $\vec{\lambda}(t_0)$, and an estimate of the distance ρ_s from the geostationary sensor to the Earth's surface (where launch occurs):

$$\vec{r}_L = \rho_s \vec{\lambda}(t_0) + \vec{R}_s(t_0)$$

For a geostationary sensor, $\vec{\lambda}(t_0)$ will generally twice intersect the surface of the Earth, depending on the altitude and latitude of the booster at t_0. The launch point may be taken as the closer of the two intersections, and ρ_s is approximated as the smaller of two real solutions of:

$$\rho_s^2 - 2R_s\rho_s \cos\vartheta + R_s^2 - R_e^2 = 0$$

where ϑ is the angle between $\vec{\lambda}(t_0)$ and $\vec{R}_s(t_0)$, and R_e is the Earth's radius. Initial conditions for the inertial velocity and thrust acceleration vectors correspond to a vertical-ascent trajectory:

$$\vec{v}_L(\vec{r}_L, \dot{r}_L) = \dot{r}_L \frac{\vec{r}_L}{|\vec{r}_L|} + \vec{\omega}_e \times \vec{r}_L, \quad \vec{a}_{c,L}(\vec{r}_L) = -k\vec{g}(\vec{r}_L)$$

where a small radial component of launch velocity $\left(\dot{r}_L = 10\,{}^{m}\!/\!{}_{s}\right)$ and a vertical excess acceleration $(k = 2.5)$ are assumed. The vertical ascent trajectory is a reasonably accurate approximation when the first observation occurs while the booster flight-path angle is greater than 60°.

A covariance matrix M_L may be determined at launch by Monte-Carlo simulation of the preceding identities that define $\vec{r}_L, \vec{v}_L, \vec{a}_{c,L}$. Errors in $\vec{\lambda}(t_0)$ are also randomized using the measurement accuracies of the sensor. Errors in radial velocity and radial acceleration are taken from a uniform distribution, with variances $\frac{1}{12}\dot{r}_L^2$ and $\frac{1}{12}\left((k-1)g\right)^2$, respectively.

The proper correlation statistics are established for $M(t_0)$ after $\vec{x}_L, M_L$ are propagated from launch at t_L to the first observation at t_0 using the nonlinear dynamical model (Table 1). Although t_L is unknown, the time interval $t_0 - t_L$ may be determined from an approximate altitude of the first observation (e.g., the cloud tops) and with a template-matching process.[14]

V. Detection and Estimation of Thrust Termination

A technique is described for detection of the final thrust termination event and for estimation of burnout time. Earlier staging events momentarily perturb the filter estimates because these transient events occur quickly. However, the final event can significantly degrade the accuracy of filter predictions at impact or at other downrange sensors.

Thrust termination can be detected from the divergence of the measurement residuals of the tracking filter. This *metric detection* technique complements other non-metric techniques such as the detection of an abrupt drop in intensity of the booster exhaust plume. The following discussion assumes that the boost sensor can observe the thrust tailoff process and a short segment of post-boost flight. As $\vec{a}_c(t) \to 0$, the time sequence of residuals $\vec{\varepsilon}_n$ exhibits a quadratic time divergence because estimates $\hat{\vec{a}}_c$ lag the actual time decay $\vec{a}_c(t)$. Thrust termination is detected when the statistical distance metric exceeds a certain threshhold:

$$\sqrt{\vec{\varepsilon}_n^{\,T} N_n^{-1} \vec{\varepsilon}_n} > \kappa$$

where κ is an integer. This detection threshhold must be tuned for reliable performance.

When thrust termination has been detected, an estimate of the burnout time t^* may be determined by processing the time sequence of residuals with a constant-gain kinematic filter.

For clarity of presentation, this kinematic filter is formulated for a scalar residual ε_n (i.e., the azimuth or elevation component of $\vec{\varepsilon}_n$), and the method is easily generalized to the vector residual $\vec{\varepsilon}_n$. Smoothed estimates of position $\hat{\varepsilon}_n$, velocity $\hat{\dot{\varepsilon}}_n$ and acceleration $\hat{\ddot{\varepsilon}}_n$ are generated with a three-state linear filter with constant gain L:

$$\hat{\vec{\xi}}_n = \overline{\vec{\xi}}_n + L(\varepsilon_n - \overline{\varepsilon}_n)$$

$$\overline{\vec{\xi}}_{n+1} = A\overline{\vec{\xi}}_n$$

$$\hat{\vec{\xi}}_n = \begin{bmatrix} \hat{\varepsilon}_n \\ \hat{\dot{\varepsilon}}_n \\ \hat{\ddot{\varepsilon}}_n \end{bmatrix}, \quad \overline{\vec{\xi}}_n = \begin{bmatrix} \overline{\varepsilon}_n \\ \overline{\dot{\varepsilon}}_n \\ \overline{\ddot{\varepsilon}}_n \end{bmatrix}, \quad A = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

where T is the sampling interval of the residuals. The residual estimate $\hat{\vec{\xi}}_n$ generated by this kinematic filter should not be confused with the full state estimate $\hat{\vec{x}}_n$ generated by the nonlinear boost filter. Constant elements of L are determined by specifying the closed-loop eigenvalues of the kinematic filter. These eigenvalues should be selected carefully to smooth ε_n because estimates of burnout time can be unreliable when the filter bandwidth is too large.

Thrust termination is assumed to cause a constant, but unknown, acceleration error $\ddot{\varepsilon}^*$ that generates a quadratic divergence in the kinematic estimates:

$$\hat{\ddot{\varepsilon}}_n = \ddot{\varepsilon}^*, \quad \hat{\dot{\varepsilon}}_n = \dot{\varepsilon}^* + \ddot{\varepsilon}^*(t_n - t^*), \quad \hat{\varepsilon}_n = \varepsilon^* + \dot{\varepsilon}^*(t_n - t^*) + \tfrac{1}{2}\ddot{\varepsilon}^*(t_n - t^*)^2$$

where $\dot{\varepsilon}^*, \varepsilon^*$ are also unknown. After some algebra, two of three unknowns may be eliminated:

$$\varepsilon^* = \hat{\varepsilon}_n - \hat{\dot{\varepsilon}}_n(t_n - t^*) + \tfrac{1}{2}\hat{\ddot{\varepsilon}}_n(t_n - t^*)^2$$

Although ε^* is unknown, it is assumed that $E(\varepsilon^*) = 0$ because the nonlinear boost filter generates optimal estimates prior to and at thrust termination. Consequently, the elapsed time $t_n - t^*$ may be determined by the roots of a quadratic polynomial:

$$\hat{\varepsilon}_n - \hat{\dot{\varepsilon}}_n\left(t_n - t^*\right) + \tfrac{1}{2}\hat{\ddot{\varepsilon}}_n\left(t_n - t^*\right)^2 = 0$$

As two real solutions are possible, the "correct" solution must be positive because $t_n > t^*$ and the solution should be consistent with other (non-metric) indicators such as intensity data. This solution process can be repeated in order to compute a mean value of t^* on a windowed interval.

For $t > t^*$, the original time sequence of measurements should be re-processed with a modified boost filter that uses a first-order model for the decay in thrust acceleration:

$$\frac{d\vec{a}_c}{dt} = -\frac{1}{\tau_c}\vec{a}_c$$

where the time constant τ_c is an input parameter that describes the thrust-decay process. Accurate estimates of t^* and τ_c are not needed.

VI. Statistical Performance Evaluation

Accuracy and stability of the nonlinear tracking filter are determined by Monte-Carlo simulation of the EKF and a theoretical boost trajectory. The theoretical booster is a multiple degree-of-freedom dynamical system that includes models for the translation and attitude motions of a variable-mass booster with three stages, and first-order lag models for the dynamics of the propulsion, thrust-vector control and attitude-control systems. During stage 1 and stage 2 flight, this booster performs gravity-turn maneuvers, which are subject to constraints on maximum angle-of-attack and lateral acceleration. Lambert guidance[19] is performed during stage 3 flight for control of the impact point.

Observations are synthesized from the boost trajectory and from two angle-only sensors in geostationary orbits. LOS elevation and azimuth angles, measured in the topocentric frame of each sensor, are corrupted with zero-mean Gaussian white noise (biases are not included). The measurement equations and observation matrix are nonlinear functions of the state (refer to

Appendix C). Asynchronous observations are generated with sampling rates and RMS accuracies that are treated as free parameters. It was assumed that the first sensor acquires the booster approximately 15 seconds after launch at 2.5 km altitude.

Measurements from two or more satellite sensors could be collected and processed at a central facility or at distributed sites (including at the sensors themselves). Centralized processing would necessitate transmission of all sensor measurements. Distributed processing could mitigate this problem because state estimates and covariances would be exchanged. Similar issues have been addressed in multi-sensor navigation problems.[20]

The EKF generates state and covariance estimates by processing each observation sequentially, when it becomes available. Between updates, estimates and covariances are propagated by Runge-Kutta numerical integration of 54 ordinary differential equations (refer to Table 1). A representative value of exhaust velocity is assumed $\left(U = 2500\, m/s\right)$. As the launch event is not observed, the estimation process begins when an approximate state estimate at launch is determined from the first observation (as discussed earlier). This estimate, together with an *a-priori* launch covariance matrix, are propagated to the first observation. Although a transient filter response is excited at the first update, this transient decays after 5 to 10 sets of observations are processed and estimation accuracy improves sequentially with more updates.

Filter performance is assessed at the last sensor observation prior to thrust termination. Probability Distribution Functions (PDFs) are determined from 500 Monte-Carlo simulations (Figures 1-3). Errors in the estimates of position, velocity and thrust-acceleration are determined by comparing filter estimates with truth. Covariance fidelity is assessed by comparison of the actual PDF with a Gaussian PDF constructed from the ensemble-averaged eigenvalues of the filter covariance matrix. A fidelity ratio η is the ratio of the actual PDF error to the Gaussian PDF error at a specified probability level.

An important finding is that acceleration process noise stabilizes the filter estimation process and improves covariance fidelity. Without process noise, filter estimates diverge. With process noise, filter covariances and gains quickly converge to quasi-constant values. Covariance fidelity depends on the noise amplitude q_a (Figure 4). For $\sqrt{q_a} \leq 0.5\, m/s^{5/2}$, the velocity and acceleration covariance matrices under-predict the true errors in the estimates $(\eta > 1)$. As q_a increases, all covariance matrices are conservative because they over-estimate the true errors $(\eta < 1)$. For this case, numerical values $1 < \sqrt{q_a} < 2\, m/s^{5/2}$ are recommended.

Accuracies of the position and velocity estimates improve with sensor measurement accuracy and more frequent observations of the trajectory (Figures 5 and 6). Accuracy is measured by the trace of the filter covariance sub-matrix at the last observation. Sensor measurement accuracy is found to have the most significant impact on track accuracy. An order-of-magnitude improvement in track accuracy results from a roughly comparable reduction in sensor measurement uncertainty.

VII. Summary and Conclusions

A nonlinear recursive algorithm has been formulated for state vector and covariance estimation of boost trajectories. A new dynamical filter model for the thrust acceleration vector of the booster is a vector-differential equation that models propellant depletion and booster attitude motions resulting from gravity-turn maneuvers and other steering maneuvers. The new model is incorporated in an extended Kalman filter with nine state variables that describe the inertial components of position, velocity and thrust acceleration. Additional algorithms were described for filter initialization and for detection and estimation of the final thrust-termination event.

Performance of the nonlinear filter was demonstrated by Monte-Carlo simulation. It was found that the boost trajectory could be successfully triangulated with asynchronous, angle-only observations from two geostationary sensors ("stereo coverage"). Accurate three-dimensional position measurements, reasonable filter update rates and acceleration process noise were found to be crucial to filter stability and covariance fidelity. At the time of the last observation, actual errors in the estimates of position and velocity were consistent with the Gaussian statistics based on the filter covariance matrix. Performance results suggest that similar results might be achieved with a constant-gain filter instead of a variable-gain filter with process noise.

Future studies should address several important issues concerning the mechanization of this filter in a real tracking system. As three-dimensional position measurements are required, the filter will need to collect and process measurements from (at least) two angle-only sensors, using a centralized or a distributed architecture. Filter performance will be degraded by errors in the time of the first observation (when the launch event is not observed), and by improper returns-to-track association caused by clutter. Finally, post-boost sensor measurements must be available in order to assess the accuracy of metric detection and estimation of the final thrust-termination event.

APPENDIX A - Model for Thrust Acceleration Magnitude

Thrust-acceleration magnitude depends on propellant massflow rate $\dot{m}$, exhaust velocity U, and instantaneous mass $m(t)$ of the booster:

$$a_c = \frac{-\dot{m}U}{m} \qquad \left(\dot{m} < 0, a_c > 0\right)$$

Total differentiation of a_c generates a differential equation:

$$\dot{a}_c = \left[\left(\frac{\dot{m}}{m}\right)^2 - \frac{\ddot{m}}{m} - \frac{\dot{m}\dot{U}}{mU}\right]U$$

Simplification is possible at steady-state operation of the motor because U and $\dot{m}$ are approximately constants, and it follows that:

$$\dot{a}_c = \frac{a_c^2}{U} \qquad (\dot{m} = 0, \dot{U} = 0)$$

This equation describes the steady-state increase in the magnitude of thrust acceleration as propellant is depleted.

Appendix B - Model for Angular Velocity

Angular velocity may be determined from an approximate form of the equations of motion. During flight in the atmosphere, a thrusting missile is generally controlled to very small angles of attack between the vehicle longitudinal axis $\vec{e}$ and the Earth-relative velocity:

$$\vec{u} = \vec{v} - \vec{\omega}_e \times \vec{r}$$

where the atmosphere is assumed to rotate at the Earth's angular velocity $\vec{\omega}_e$. After differentiation of $\vec{u}$, the equations of motion may be re-cast in an equivalent form:

$$\frac{d\vec{u}}{dt} = \frac{d\vec{v}}{dt} - \vec{\omega}_e \times \frac{d\vec{r}}{dt} = \vec{a}_c + \vec{a}_a + \vec{g}(\vec{r}) - \vec{\omega}_e \times \vec{v}$$

Aerodynamic acceleration arises from drag acceleration $\vec{a}_D$ and lift acceleration $\vec{a}_L$:

$$\vec{a}_a = \vec{a}_D + \vec{a}_L$$

where $\vec{a}_D$ is parallel and opposite to $\vec{u}$, and $\vec{a}_L$ is perpendicular to $\vec{u}$ in the plane containing $\vec{u}$ and $\vec{e}$. The Coriolis term $\vec{\omega}_e \times \vec{v}$ is negligibly small, and may be omitted.

A kinematic identity may be used to express the time derivative with respect to inertial space in terms of a time derivative relative to a reference frame attached to $\vec{u}$:

$$\frac{d\vec{u}}{dt} = \dot{u}\left(\frac{\vec{u}}{u}\right) + \vec{\Omega} \times \vec{u}$$

In the special case when $\vec{e}$ is fixed in this rotating frame, the angular velocity $\vec{\Omega}$, as seen by the inertial observer, must be the same as the angular velocity $\vec{\omega}$ of the body frame:

$$\vec{\Omega} = \vec{\omega}$$

Collecting terms, a unique solution for $\vec{\omega}$ may be extracted from this approximate form of the equations of motion:

$$\dot{u}\left(\frac{\vec{u}}{u}\right) + \vec{\omega} \times \vec{u} = \vec{a}_c + \vec{a}_D + a_L + \vec{g}(\vec{r})$$

Terms parallel to $\vec{u}$ may be removed by a vector crossproduct with $\vec{u}$:

$$(\vec{\omega} \times \vec{u}) \times \vec{u} = \left(\vec{a}_c + \vec{a}_L\right) \times \vec{u} + \vec{g} \times \vec{u}$$

The left-hand side of this expression may be expanded using the triple-vector crossproduct identity:

$$(\vec{\omega} \times \vec{u}) \times \vec{u} = (\vec{\omega} \cdot \vec{u})\vec{u} - u^2 \vec{\omega}$$

As $\vec{\omega} \cdot \vec{u} = 0$ because the vehicle is stabilized in roll, it follows that:

$$\vec{\omega} = \frac{\vec{u} \times (\vec{a}_c + \vec{g})}{u^2}, \quad |\vec{a}_L| << |\vec{a}_c|$$

where $\vec{a}_L$ has been neglected. This approximation is justified during a gravity-turn maneuver (because the angle-of-attack is controlled to very small values) and during exoatmospheric flight (when dynamic pressure and lift acceleration are negligibly small, even at large angles of attack).

Following substitution of $\vec{g} = -\frac{\mu_e}{r^3}\vec{r}$, it is found that:

$$\vec{\omega} = \frac{1}{u^2}\left(\vec{u} \times \vec{a}_c\right) + \lambda(\vec{r} \times \vec{u}), \qquad \lambda = \frac{\mu_e}{r^3 u^2}$$

where μ_e is the Earth's gravitational parameter. The first term arises from components of $\vec{a}_c$ that are perpendicular to $\vec{u}$. The second term arises from gravitational acceleration, which usually has a component perpendicular to $\vec{u}$. In the special case of vertical ascent, these terms are in exact opposition, and it follows that $\vec{\omega} = 0$. During a gravity-turn maneuver at zero angle of attack, angular velocity arises from the gravitational term only:[16]

$$\vec{\omega} = \lambda(\vec{r} \times \vec{u}) \qquad (\vec{u} \times \vec{a}_c = 0)$$

Appendix C – Measurement Model

The boost sensor measures the angular components of the line-of-sight (LOS) vector $\vec{\rho}$ from the angle-only sensor to the booster:

$$\vec{\rho} = \vec{r} - \vec{r}_s$$

where $\vec{r}, \vec{r}_s$ are the inertial position vectors of the booster and boost sensor, respectively. For simplicity, the angular measurements are the elevation angle γ relative to the local horizontal, and the LOS azimuth angle ψ relative to North:

$$\gamma = \tan^{-1}\left(\frac{\rho_r}{\sqrt{\rho_e^2 + \rho_n^2}}\right), \quad \psi = \tan^{-1}\left(\frac{\rho_e}{\rho_n}\right)$$

where ρ_r, ρ_e, ρ_n are the radial, east and north components of $\vec{\rho}$ at the location of the sensor:

$$\begin{bmatrix} \rho_r \\ \rho_e \\ \rho_n \end{bmatrix} = \Theta(\vec{r}_s)(\vec{r} - \vec{r}_s)$$

The rotation matrix Θ, from inertial coordinates to local-level coordinates of the boost sensor, depends on the angular coordinates of $\vec{r}_s$.

The measurement $\vec{y}$ and observation matrix H are nonlinear functions of $\vec{r}$:

$$\vec{y} = \vec{h}(\vec{r}) = \begin{bmatrix} \gamma(\vec{r}) \\ \psi(\vec{r}) \end{bmatrix}$$

$$H(\vec{r}) = \frac{\partial \vec{h}}{\partial(\vec{r}^T, \vec{v}^T, \vec{a}_c^T)} = \begin{bmatrix} \dfrac{\partial(\gamma, \psi)}{\partial(r_1, r_2, r_3)} & O_{2\times 6} \end{bmatrix}$$

Referring to H, the zero matrix $O_{2\times 6}$ appears because $\vec{h}$ is independent of velocity $\vec{v}$ and thrust acceleration $\vec{a}_c$. The Jacobian matrix may be determined by the chain rule for partial derivatives:

$$\frac{\partial(\gamma,\psi)}{\partial(r_1,r_2,r_3)} = \frac{\partial(\gamma,\psi)}{\partial(\rho_r,\rho_e,\rho_n)} \frac{\partial(\rho_r,\rho_e,\rho_n)}{\partial(r_1,r_2,r_3)} = \frac{\partial(\gamma,\psi)}{\partial(\rho_r,\rho_e,\rho_n)} \Theta(\vec{r}_s)$$

Following partial differentiations of the expressions for γ and ψ, it may be shown that:

$$\frac{\partial(\gamma,\psi)}{\partial(\rho_r,\rho_e,\rho_n)} = \begin{bmatrix} \frac{\sqrt{\rho_e^2+\rho_n^2}}{\rho^2} & \frac{-\rho_e\rho_r}{\rho^2\sqrt{\rho_e^2+\rho_n^2}} & \frac{-\rho_n\rho_r}{\rho^2\sqrt{\rho_e^2+\rho_n^2}} \\ 0 & \frac{\rho_n}{\rho_e^2+\rho_n^2} & \frac{-\rho_e}{\rho_e^2+\rho_n^2} \end{bmatrix}$$

$$\rho = \sqrt{\rho_r^2+\rho_e^2+\rho_n^2}$$

Collecting terms, the observation matrix is specified by:

$$H(\vec{x}) = \begin{bmatrix} \begin{bmatrix} \frac{\sqrt{\rho_e^2+\rho_n^2}}{\rho^2} & \frac{-\rho_e\rho_r}{\rho^2\sqrt{\rho_e^2+\rho_n^2}} & \frac{-\rho_n\rho_r}{\rho^2\sqrt{\rho_e^2+\rho_n^2}} \\ 0 & \frac{\rho_n}{\rho_e^2+\rho_n^2} & \frac{-\rho_e}{\rho_e^2+\rho_n^2} \end{bmatrix} \Theta(\vec{r}_s) & O_{2\times 6} \end{bmatrix}$$

where the functions $\rho_r(\vec{r}), \rho_e(\vec{r}), \rho_n(\vec{r})$ are specified implicitly by the preceding identities.

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