Evaluation & Reduction of Multipath-Induced Bias on GPS Time-of-Arrival

by

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Abstract

This paper presents new expressions for the multipath-induced pseudorange error(i.e. bias) and variance introduced by multipath onto the time of arrival estimate obtained using a non-coherent early-late gate discriminator. The results include the effect of front-end bandwidth and early-late gate spacing.

We also investigate a blind method for cancelling the multipath, in order to improve the time-of-arrival estimate. Our approach uses early-late gate processing on an objective function derived from an adaptive FIR filter that attempts to match the crosscorrelation of the received signal with a multipath-free replica of the desired crosscorrelation. This method performs reasonably well, and decreases the multipath-induced pseudorange error by approximately a factor of 2, even in very stressing multipath environments.

Acknowledgment

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Introduction

The Global Positioning System (GPS) uses the measured time delays between a user and at least four satellites to estimate user position [1, 2]. Unfortunately, multipath can introduce [3-5] a bias in the measured time delay that cannot be removed by smoothing or by narrow-correlator receivers[6]. Advances in receiver technology are producing receivers with wider front-end bandwidths and narrower early-late gate spacings; which helps to mitigate multipath effects.

In this paper we will investigate an approach to cancel the multipath. As part of the analysis, we will first obtain new results for the multipath-induced pseudorange error and variance in the time of arrival estimate obtained by an early-late gate discriminator. These results include the effects of both front-end bandwidth and gate spacing. The multipath cancellation approach we consider for multipath equalization differs from (1) the classical approach, which is to estimate and invert the channel, and (2) various maximum-likelihood-estimation approaches [7-18]. The maximum likelihood method is very powerful, but is computationally complex, especially when there are many multipaths and the number of multipaths is unknown.That is,the maximum likelihood method postulates there is a direct path of unknown complex amplitude and delay,plus M multipaths with unknown complex amplitudes and delays.These unknown parameters are then all simultaneously estimated using the measured data. Furthermore, some of the algorithms (e.g., Expectation Maximization) used to estimate the multipath parameters sometimes converge to spurious estimates,although the strobe correlator[12] appears to offer significant benefit. Consequently, we consider a blind equalization approach that simultaneously cancels multipath and estimates the true time delay. No information about the channel is required, other than the assumption that the coherent sum of the multipath signals is weaker than the direct-path signal. We will demonstrate that for a given front-end bandwidth and gate spacing, our method can reduce the bias by a factor of two (or more) relative to a conventional noncoherent early-late gate discriminator, even in stressing multipath environments.

Bias and Variance Due to Multipath

We first wish to obtain formal expressions for the multipath-induced bias and variance in the time-of-arrival estimate obtained using noncoherent early-late gate processing. A noncoherent early-late gate processor estimates time-of-arrival by minimizing the error function(also called the delay-lock loop tracking error)

$$e(\hat{\tau}) = F\left(\hat{\tau} + \delta/2\right) - F\left(\hat{\tau} - \delta/2\right) \tag{1}$$

where $\hat{\tau}$ is the estimate of the true delay τ, δ is the time interval between the early and late gates, $F(\tau) = |r_v(\tau)|^2$ and $r_v(\hat{\tau})$ is the crosscorrelation defined as

$$r_v\left(\hat{\tau}\right) = \frac{1}{T_o}\int\ dt\ v^*\left(t\right)s\left(t-\hat{\tau}\right) \tag{2}$$

In Equation (2), T_o is the integration time of the correlator,* denotes complex conjugate, *s(t)* is the transmitted signal and *v(t)* is the received voltage, which can be modeled as

$$v\left(t\right) = s\left(t-\tau\right) + \sum_{p=1}^{Q} a_p s\left(t-\tau-\tau_p\right) + x\left(t\right) \tag{3}$$

where $s(t - \tau)$ is the direct-path signal, τ is the true direct-path delay, *x(t)* is the noise, a_p, τ_p are, respectively, the complex amplitude and differential delay of multipath component *p* and *Q* is the number of multipath scattering centers.One of the multipaths could represent a ground wave,if one exists. In writing Equation (3) we have assumed that the differential Doppler shifts between the multipath and direct-path signal are small. This assumption is valid in most instances,but there are cases[19] where it fails. If Equation (3) is used in (2) we obtain

$$\begin{aligned} r_v\left(\hat{\tau} \pm \delta/2\right) = \sum_{p=o}^{Q} \frac{a_p^*}{T_o}\int_o^{T_o} dt\ s^*\left(t-\tau-\tau_p\right)s\left(t-\hat{\tau}\mp\delta/2\right) \\ + \xi\left(\pm\delta/2\right) \end{aligned} \tag{4}$$

where we have defined $a_o \equiv 1, \tau_o \equiv 0$, and the noise term ξ is

$$\xi\left(\pm\delta/2\right) = \frac{1}{T_o}\int_o^{T_o} dt\ x^*\left(t\right)s\left(t-\hat{\tau}\mp\delta/2\right)\ . \tag{5}$$

Let us assume that the received signal has been first filtered by an ideal filter of bandwidth *B*. Then we can express *s(t)* in terms of its Fourier transform *S(f)* as

$$s\left(t\right) = \int_{-B/2}^{B/2} df\ S\left(f\right)\exp\left(i2\pi ft\right)\ . \tag{6}$$

If Equation (6) is used in (4), and we use the fact that the correlation interval T_o is very much larger than any errors in delay, we can rewrite Equation (4) as

$$\begin{aligned} r_v\left(\hat{\tau} \pm \delta/2\right) = C_s\sum_{p=o}^{Q} a_p^*\int_{-B/2}^{B/2} df P\left(f\right)\cos\left[2\pi f\left(\varepsilon\mp\delta/2+\tau_p\right)\right] \\ + \xi\left(\pm\delta/2\right) \end{aligned} \tag{7}$$

where *P(f)* is the normalized* power spectrum of the signal, defined as

$$C_s P\left(f\right) = \frac{\left|S\left(f\right)\right|^2}{T_o}\ , \tag{8}$$

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* *P(f)* is normalized such that $\int_{-\infty}^{\infty} P(f)df = 1$.

C_s is the signal carrier power, and

$$\varepsilon = \tau - \hat{\tau} \quad . \tag{9}$$

In writing Equation (7) it has been assumed that *P(f)* is a symmetric function of frequency, *f*. Although the results are valid for arbitrary symmetric power spectra, in this paper *P(f)* will be chosen as the pseudonoise (PN) power spectrum

$$P(f) = T_C \operatorname{sinc}^2\left(\pi f T_C\right) \tag{10}$$

where T_C is the chip duration of the PN signal.

Now substitute Equation (7) into (1), and assume[*] that $\left|\tau_p\right| \le T_c$ and $B\varepsilon < 0.25$ so in Equation (7)

$$\cos\left[2\pi f\left(\varepsilon \mp \delta/2 + \tau_p\right)\right] = \cos(2\pi f\varepsilon)\cos\left[2\pi f\left(-\tau_p \pm \delta/2\right)\right] + \sin(2\pi f\varepsilon)\sin\left[2\pi f\left(-\tau_p \pm \delta/2\right)\right]$$
$$\simeq \cos\left[2\pi f\left(-\tau_p \pm \delta/2\right)\right] + 2\pi f\,\varepsilon \sin\left[2\pi f\left(-\tau_p \pm \delta/2\right)\right],$$

where the last step follows by using the Taylor series approximation for $\cos(2\pi f\varepsilon)$ and $\sin(2\pi f\varepsilon)$. Then, we obtain for the error function defined in Equation (1), the result

$$e(\hat{\tau}) = K_o\varepsilon + b + \rho \tag{11}$$

where K_o and b are independent of $\varepsilon = \tau - \hat{\tau}$, and are defined as

$$K_o = 4\pi C_s^2 \sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p a_q^* \left[X_p Y_q + Y_p X_q - Z_p H_q - H_p Z_q\right] , \tag{12}$$

$$b = -2C_s^2 \sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p a_q^* \left[X_p Z_q + Z_p X_q\right] , \tag{13}$$

$$X_p = \int_{-B/2}^{B/2} df\, P(f)\cos(\pi f\delta)\cos\left(2\pi f\tau_p\right) , \tag{14}$$

$$Y_q = \int_{-B/2}^{B/2} df\, P(f)\, f \sin(\pi f\delta)\cos\left(2\pi f\tau_q\right) , \tag{15}$$

$$Z_p = -\int_{-B/2}^{B/2} df\, P(f)\sin(\pi f\delta)\sin\left(2\pi f\tau_p\right) , \tag{16}$$

$$H_q = -\int_{-B/2}^{B/2} df\, P(f)\, f \cos(\pi f\delta)\sin\left(2\pi f\tau_q\right) , \tag{17}$$

and ρ is a noise term. If noise is ignored, and we set the error equal to zero, we find

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[*] Although the approximation used here is valid for the power spectrum in Equation (10) provided $B\varepsilon < 0.25$, we caution that the requirement is much more stringent for power spectra, such as binary offset carrier (BOC), that have much of their energy concentrated near the band edges. In fact, we have found that for the BOC (10, 5) that is being contemplated for the new military (M) code the approximation is inadequate, and a numerical evaluation of Equation (1) is required to accurately calculate the bias error introduced by multipath.

$$\varepsilon = -\frac{b}{K_o} \tag{18}$$

or substituting for ε

$$\hat{\tau} = \tau + \frac{b}{K_o} \quad . \tag{19}$$

Consequently, the estimate $\hat{\tau}$ of the true delay τ is biased, and that bias is equal to b/K_o. If multipath is absent altogether then $a_o = 1$ and all other $a_p = 0$. Then, because $\tau_o = 0$, by definition, it is readily shown that $b = 0$, so that the bias vanishes, as expected.

In order to validate Equation (18) we considered a single multipath $(a_1 = 0.5)$ and evaluated the integrals in Equations (12) to (17) with $BT_c = 8$ for the limits when the multipath is in-phase and out of phase with the direct path. These results are presented in figure 1 and compared with the theoretical results for $B \to \infty$ in reference 5. Note that the agreement is quite good.

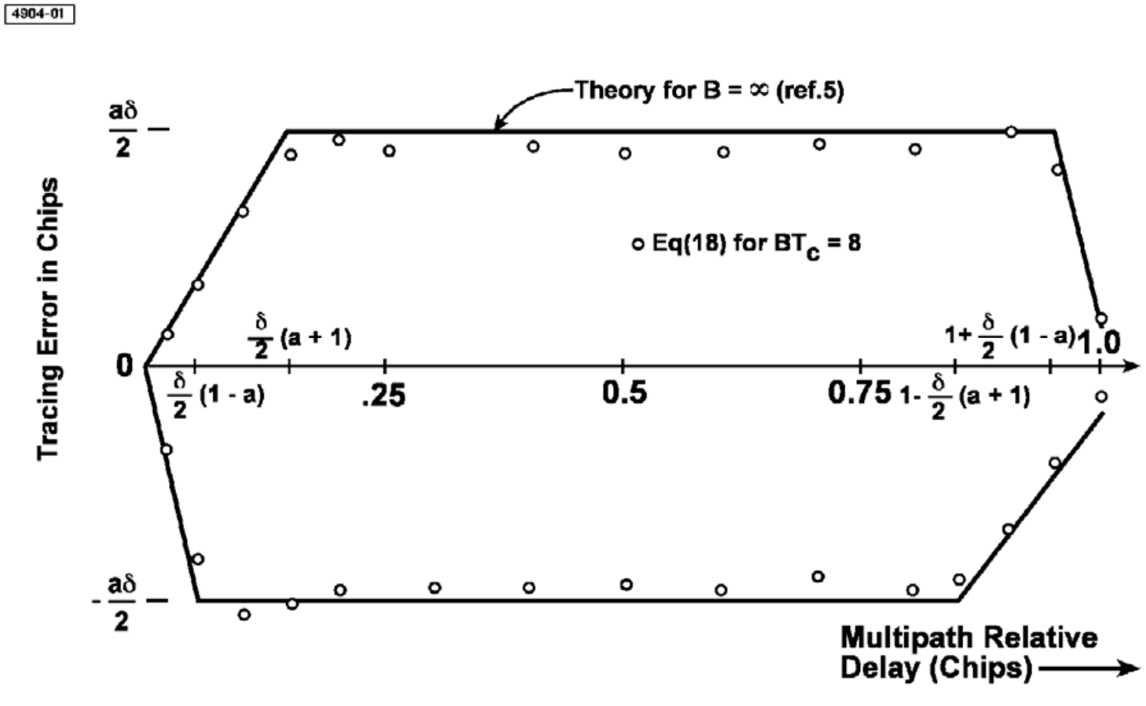

Figure 1. Multipath-induced Tracking error for $\delta = 0.2T_c$ as computed From Equation (18) when $BT_c = 8$

We now derive the variance of the time-of-arrival estimate $\hat{\tau}$. Let us first define

$$D(\theta) = C_s \int_{-B/2}^{B/2} df\, P(f) \cos 2\pi f\theta \, . \tag{20}$$

Then, Equation (7) can be rewritten as

$$r_v(\hat{\tau} \pm \delta/2) = \sum_{p=o}^{Q} a_p^* D(\varepsilon \mp \delta/2 + \tau_p) + \xi(\pm \delta/2) \tag{21}$$

where ξ is defined in Equation (5). Therefore, the error $e(\hat{\tau})$ defined in Equation (1) is

$$\begin{aligned} e(\hat{\tau}) &= F(\hat{\tau} + \delta/2) - F(\hat{\tau} - \delta/2) \\ &= \sum_{p=0}^{Q} \sum_{q=0}^{Q} a_p a_q^* \Lambda_{pq} \\ &+ \sum_{p=0}^{Q} a_p \left[D(\varepsilon + \tau_p - \delta/2) \xi(\delta/2) - D(\varepsilon + \tau_p + \delta/2) \xi(-\delta/2) \right] \\ &+ \sum_{q=0}^{Q} a_q^* \left[D(\varepsilon + \tau_q - \delta/2) \xi^*(\delta/2) - D(\varepsilon + \tau_q + \delta/2) \xi^*(-\delta/2) \right] \\ &+ \left[|\xi(\delta/2)|^2 - |\xi(-\delta/2)|^2 \right] \end{aligned} \tag{22}$$

where

$$\Lambda_{pq} = D(\varepsilon + \tau_p - \delta/2) D(\varepsilon + \tau_q - \delta/2) - D(\varepsilon + \tau_p + \delta/2) D(\varepsilon + \tau_q + \delta/2) \quad . \tag{23}$$

The first term on the right-hand side of Equation (22) reduces to $K_o \varepsilon + b$, in the limit when $\varepsilon B << 1$, where K_o and b are defined in Equations (12) and (13) respectively. The last three terms on the right-hand side of Equation (22) have been denoted by ρ in Equation (11), and represent the effects of the noise. In the limit when ε is small these noise terms can be approximated by

$$\begin{aligned} \rho &= \sum_{p=0}^{Q} a_p \left[D(-\tau_p + \delta/2) \xi(\delta/2) - D(\tau_p + \delta/2) \xi(-\delta/2) \right] \\ &+ \sum_{q=o}^{Q} a_q^* \left[D(-\tau_q + \delta/2) \xi^*(\delta/2) - D(\tau_q + \delta/2) \xi^*(-\delta/2) \right] \\ &+ \left[|\xi(\delta/2)|^2 - |\xi(-\delta/2)|^2 \right] \end{aligned} \tag{24}$$

where we have used the fact that $D(-\theta) = D(\theta)$.

We first need to calculate the variance of ρ defined as

$$\sigma_\rho^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2 \, . \tag{25}$$

Then, upon referring to Equation (11), the variance in ε and hence $\hat{\tau}$, is given by

$$\sigma_{\hat{\tau}}^2 = \frac{\sigma_\rho^2}{K_o^2} \tag{26}$$

where K_o is defined in Equation (12). Let us assume that the noise is a circular[*], gaussian random process with a symmetric power spectrum $P_{xx}(f)$. Then, after considerable manipulation, one can show that

$$\begin{aligned}\sigma_\rho^2 = \frac{2}{T_o}\sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p a_q^* \Big\{ \big[& D(\tau_p - \delta/2) D(\tau_q - \delta/2) \\ & + D(\tau_p - \delta/2) D(\tau_q - \delta/2)\big] J(0) - D(\tau_p - \delta/2) D(\tau_q + \delta/2) J(\delta) \\ & - D(\tau_p + \delta/2) D(\tau_q - \delta/2) J(\delta)\Big\} \\ & + \frac{2}{T_o^2}\left[J^2(0) - J^2(\delta)\right].\end{aligned} \tag{27}$$

where

$$J(\theta) = \int_{-B/2}^{B/2} df\, P(f) P_{xx}(f) \cos 2\pi f\theta\,. \tag{28}$$

Equation (27) is lengthy, but we can verify its correctness by taking the limiting case when the multipath is absent. Then only the $p=q=0$ term remains in Equation (27), and K_o and σ_ρ^2 reduce to

$$K_o = 8\pi C_S^2 \int_{-B/2}^{B/2} df\, P(f) \cos(\pi f \delta) \int_{-B/2}^{B/2} df' f' P(f') \sin(\pi f' \delta) \tag{29}$$

$$\begin{aligned}\sigma_\rho^2 = & \frac{8C_S^3}{T_o}\left(\int_{-B/2}^{B/2} df P(f) \cos \pi f \delta\right)^2 \int_{-B/2}^{B/2} df\, P(f) P_{xx}(f) \sin^2 \pi f \delta \\ & + \frac{2C_S^2}{T_o^2}\left\{\left(\int_{-B/2}^{B/2} df P(f) P_{xx}(f)\right)^2 - \left(\int_{-B/2}^{B/2} df P(f) P_{xx}(f) \cos \pi f \delta\right)^2\right\}\end{aligned} \tag{30}$$

and

$$\sigma_{\hat{\tau}}^2 = \frac{\sigma_\rho^2}{K_o^2} \quad . \tag{31}$$

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[*] A circular, gaussian random process has the property that $\left\langle X\ X^T \right\rangle = 0$, whereas $\left\langle X\ X^H \right\rangle \neq 0$, where the superscript H denotes conjugate transposed and $\langle \cdots \rangle$ denotes an expectation.

This result is in exact agreement with the results obtained in References 20 and 21, for the case when $P_{xx}(f)$ is symmetric in f (as we have assumed).

It is useful to normalize the variance to the Cramér-Rao bound, which for white noise and no multipath is[20]

$$\sigma_{CR}^2 = \frac{1}{8\pi^2\left(\frac{C_S T_o}{N_o}\right)\beta} \tag{32}$$

where N_o is the power spectral density of the white noise, and

$$\beta = \int_{-B/2}^{B/2} df\, f^2 P(f). \tag{33}$$

We now assume that the signal power spectrum is that for a pseudonoise (PN) sequence, given by Equation (10). For this signal, the normalized Cramér-Rao bound in Equation (28) is

$$\frac{\sigma_{CR}}{T_C} = \frac{\alpha}{\left(\frac{C_S T_o}{N_o}\right)^{1/2}} \tag{34}$$

where α is given in Table 1.

Table 1. Values of α for Different Bandwidths

BT_C	α
1	.5
2	.35
3	.29
4	.25
5	.22
6	.20
7	.19

The Cramér-Rao bound in Equations (32) and (34) is the unsmoothed error. In refs. 20 and 21 it is demonstrated that the Cramér-Rao bound, after smoothing by a tracking loop of bandwidth B_L, is approximately

$$\sigma_{CRS}^2 = 2B_L T_o \sigma_{CR}^2 \tag{35}$$

so that the smoothed error can be written as

$$\frac{\sigma_{CRS}}{T_C} = \alpha \left(\frac{2B_L}{C_S / N_o} \right)^{1/2} . \tag{36}$$

For *P(Y)* code the average value of C_S/N_o at zenith[*] for a receiver with a noise figure of 4 dB and a 3 dB antenna gain is approximately 44 dB-Hz. Also, the loop bandwidth B_L can range from slightly less than 1 Hz up to 10 Hz. For B_L = 4 Hz, C_S/N_o = 44 dB-Hz and BT_C = 3 we find from Equation (36) that

$$\frac{\sigma_{CRS}}{T_C} = 0.005 \quad . \tag{37}$$

Let us now present some quantitative results on how multipath affects the bias and standard deviation of the time-of-arrival estimate. We assume that there is a single multipath scatterer of strength $a_1 = 0.5\exp(i\phi)$ where ϕ is a random variable that is uniformly distributed between $-\pi$ and π. The delay τ_1 (relative to the direct-path delay) is chosen as $\tau_1 = (0.2 + 0.8\eta)T_C$ where η is a random variable that is uniformly distributed between 0 and 1. We then used Equations (19) and (26) to calculate the bias and standard deviation (normalized to the Cramer-Rao bound) for 100 random draws of the variables (ϕ, η). Figure 2 shows the average of the absolute values of the bias for the case when the gate spacing $\delta = 0.1\, T_C$. We also performed calculations for other values of δ. For $\delta < 0.2T_c$ the results were rather insensitive to the value of δ, but for $\delta > 0.2T_c$ the bias increased as δ was increased. From Figure 2 we observe that the bias error is significant, but, as is well known, can be reduced by increasing the front-end bandwidth *B*. We also calculated the bias when two and three multipaths are present, and as expected, the bias error was larger than that shown in Figure 2, but still decreased as *B* was increased.

In Figure 3 we show how multipath affects the normalized standard deviation of the time-of-arrival estimate. Again, each point is the average of 100 random draws of (ϕ, η). From Figure 3 we observe that the multipath does not drastically increase the standard deviation, because the results are only slightly larger than the Cramér-Rao bound given by Equation (34).

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[*] That is, $C/N_o = P + G - 10\log_{10}(kT_o) - N_f - L \quad dB - Hz,$ where P is the average received power for $P(Y)$ code = –157.2 dBW (the average is approximately 5.4dB above the minimum value of –162.6 dBW), G is the antenna gain at zenith = 3dB, R is Boltzmann's constant, $T_o = 293^{\circ}\,\mathrm{K}, \mathrm{N_f} =$ noise figure = 4dB and L = loss = 2dB.

Because it is not always feasible to use large front-end bandwidths to reduce the multipath-induced bias, we now examine a simple approach to further reduce bias by multipath equalization.

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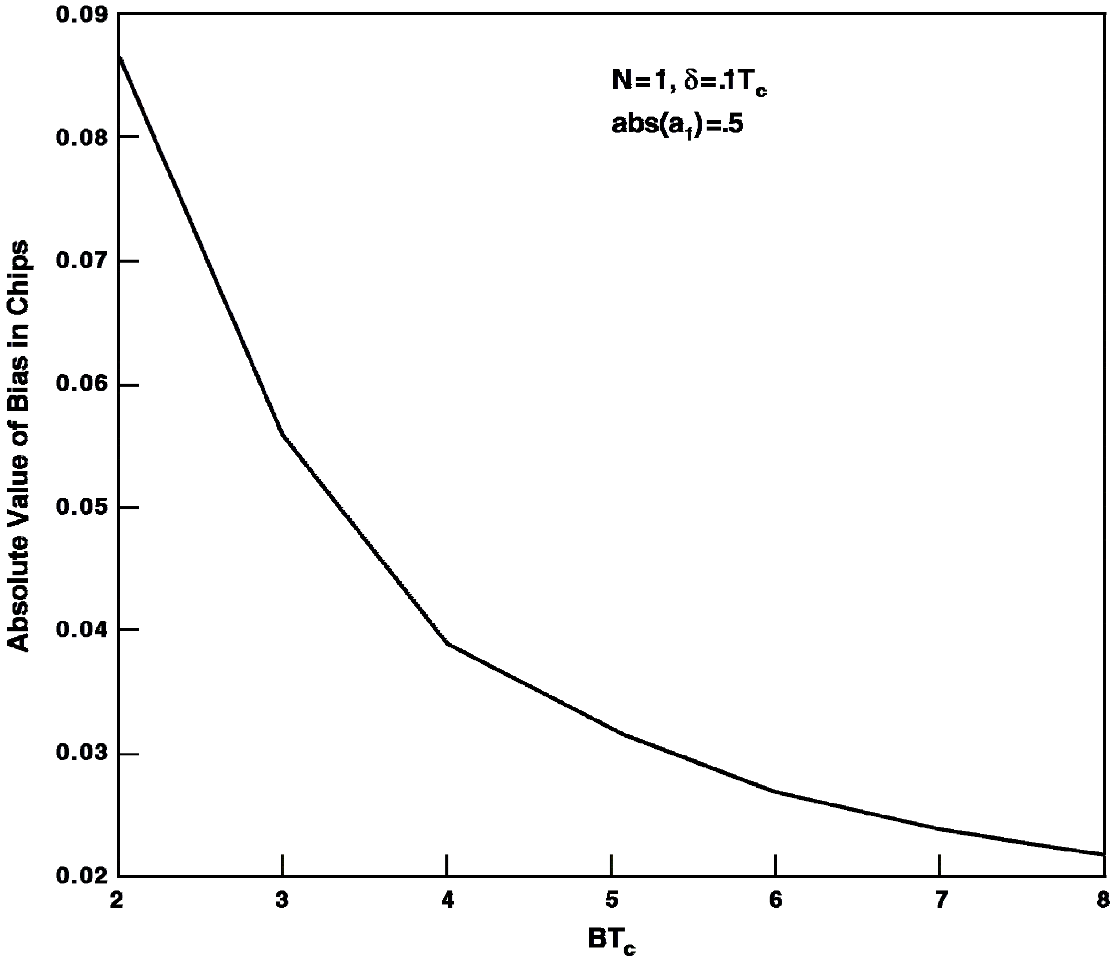

Figure 2. Effect of Input Bandwidth on Bias for 1 Multipath

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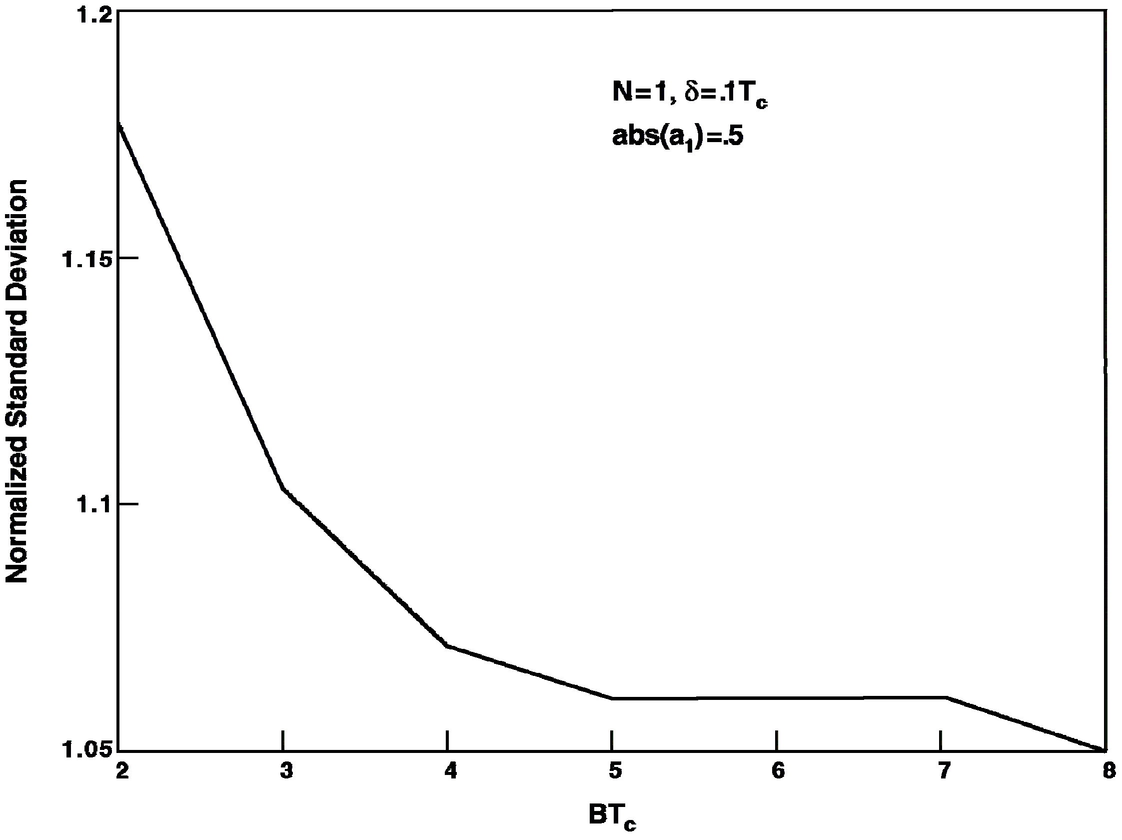

Figure 3. Effect of Input Bandwidth on Standard Deviation for 1 Multipath

Multipath Equalization

We wish to use a finite impulse response (FIR) filter to reduce the bias that is introduced by multipath on the estimate of the direct path time delay. The FIR filter is shown in Figure 4, and its unknown weights are estimated using the approach shown conceptually in Figure 5. The overall method proceeds in two steps. First, we postulate that the direct-path signal has a delay $\hat{\tau}$, so that the received signal is $s(t-\hat{\tau})$ with a corresponding conditional[*] crosscorrelation function $R(\theta-\hat{\tau})$. The actual received voltage *v(t)* produces a voltage *y(t)* at the output of the equalization filter in Figure 5, and has a corresponding crosscorrelation function $R_y(\theta)$. The unknown weights in the equalization filter are then determined by trying to match $R_y(\theta)$ to the desired conditional correlation function $R(\theta-\hat{\tau})$. In particular, we determine the weights in the FIR filter by minimizing the conditional error $E(\hat{\tau})$, defined as the mean square difference between

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[*] i.e., conditional on the assumed value $\hat{\tau}$

$R(\theta-\hat{\tau})$ and $R_y(\theta)$. Once the filter weights for the postulated delay $\hat{\tau}$ are known, these are used in $E(\hat{\tau})$, and the second step is to search the conditional error function $E(\hat{\tau})$ for that value τ that produces a minimum. The value τ is then declared to be the estimate of direct-path delay.

Now let us explain in detail how the approach works. Let *v(t)* be the total received voltage consisting of the direct-path signal plus multipath and noise. Then, referring to Figure 4, the output of the FIR filter is

$$\begin{aligned} y(t) &= \sum_{k=-K}^{K} w_k v(t+kT) \\ &= W^T V \end{aligned} \tag{38}$$

where $W^T = [w_{-K} \cdots w_o \cdots w_K], V^T = [v(t-KT) \cdots v(t+KT)]$ and *N = 2K+1* is the total number of time taps in the FIR filter. Note that *N = 1 (K = 0)* corresponds to the case when the filter is absent. The number of taps must be such that *KT* is of order of the largest expected multipath delay. The unknown weights w_k will be calculated shortly. By using Equation (38) it is readily seen that the output $R_y(\theta)$ of the crosscorrelator is given by

$$R_y(\theta) = \sum_{n=-K}^{K} w_n \frac{1}{T_o} \int_o^{T_o} dt\, s^*(t-\theta) v(t+nT) \tag{39}$$

where *K* is assumed to be sufficiently large so that *KT* is at least equal to the largest multipath delay relative to the direct path.

If we recall the definition of the crosscorrelation function of the received voltage in Equation (2) it is readily seen that Equation (39) can be rewritten as

$$R_y(\theta) = \sum_{n=-K}^{K} w_n\, r_v(\theta+nT) \quad . \tag{40}$$

Now, define the vector

$$R_v = [r_v(\theta-KT) \cdots r_v(\theta+KT)]^T \quad . \tag{41}$$

Then, Equation (40) can be rewritten as

$$R_y(\theta) = W^T R_v = R_v^T W \quad . \tag{42}$$

We choose the weight vector *W* and the time delay estimate $\hat{\tau}$ to minimize the error function

$$E(\hat{\tau}) = \frac{1}{2T_1} \int_{\hat{\tau}-T_1}^{\hat{\tau}+T_1} \left| R(\theta-\hat{\tau}) - R_y(\theta) \right|^2 d\theta \tag{43}$$

where $R(\theta)$ is the desired signal crosscorrelation function. This minimization proceeds in two steps: We first assume $\hat{\tau}$ is the correct delay and then choose the weight vector *W* that minimizes *E* given $\hat{\tau}$. Then, once *W* is known we substitute that *W* into Equation (42) and then minimize *E* with respect to $\hat{\tau}$.

If we substitute Equation (42) into (43), and then minimize with respect to *W,* we find that

$$W = \Lambda^{-1} C \tag{44}$$

where the *(2K+1)x1* vector *C* and the *(2K+1)x(2K+1)* matrix *Λ* are defined as

$$C = \frac{1}{2T_1} \int_{\hat{\tau}-T_1}^{\hat{\tau}+T_1} d\theta \, R_v^*(\theta) R(\theta - \hat{\tau}), \tag{45}$$

$$\Lambda = \frac{1}{2T_1} \int_{-T_1}^{T_1} d\theta \, R_v^*(\theta) R_v(\theta)^T \quad . \tag{46}$$

If Equation (44) is substituted into (43) we obtain

$$E(\hat{\tau}) = \frac{1}{2T_1} \int_{-T_1}^{T_1} |R(\theta)|^2 d\theta - C^H \Lambda^{-1} C \quad . \tag{47}$$

Because the first term on the right-hand side of Equation (47) is a positive definite quantity it is clear that *E* is minimized by the delay τ that maximizes the second term. Therefore,

$$\tau = \arg\max_{\hat{\tau}} \left[C^H \Lambda^{-1} C \right] \quad . \tag{48}$$

Note that there are multiple maxima in the function $C^H \Lambda^{-1} C$ for $K > 0$. Therefore, the conventional *(K=0)* solution is always used to resolve the ambiguity.

Observe that, in practice, the weight vector *W* is never actually calculated nor is there a real FIR filter. These are artifices used to obtain the result in Equation (48). Rather all one needs to do is to calculate the crosscorrelation functions in Equations (45) and (46), so that τ can be estimated.

The peak of the function

$$Q_o(\hat{\tau}) = C^H \Lambda^{-1} C \tag{49}$$

can be estimated by non-coherent early-late gate processing, using an error function

$$e(\hat{\tau}) = Q_o(\hat{\tau} + \delta/2) - Q_o(\hat{\tau} - \delta/2) \tag{50}$$

where $\hat{\tau}$ is the estimate of the true time delay τ and δ is the time interval between the early and late gates. In the Appendix we derive an analytical expression for the bias error obtained with this approach. This error can be expressed as b_1/K_1, where b_1 and K_1 are presented in Equations (A15) and (A16) of the Appendix.

Let us now examine the bias error reduction that can be achieved using the aforementioned approach. In order to study this problem we choose an early-late gate spacing $\delta = 0.1\ T_C$, a signal-to-noise ratio (after integration gain) of 20 dB and first assume that only a single multipath scatterer is present with complex amplitude

$$a_1 = 0.5 \exp(i\phi_1) \tag{51}$$

and delay

$$\frac{\tau_1}{T_C} = \eta_1 \tag{52}$$

where ϕ_1 is uniformly distributed random variable $(0, 2\pi)$ and η_1 is uniformly distributed $(0, 1)$. The multipath-induced bias error both with (N = 5) and without the multipath cancellation filter is shown in Figure 6, as a function of front-end bandwidth, for the case when the tap spacing* $T = 0.5\ T_C$. Each point is the average of 200 Monte Carloes over the random parameters (ϕ_1, η_1). Note that for $BT_C = 3$ the filter decreases the bias by more than a factor of 2.5 but the decrease is only about 2 for $BT_C = 5$. Nevertheless, the filter always decreased the multipath-induced bias, relative to the conventional, non-coherent early-late gate discriminator. Although we have chosen a multipath amplitude of 0.5 for this example, the method still works for smaller amplitude multipaths, but the improvement is not as large(because the bias is already very small).

Next, consider two multipaths with the parameters for multipath 1 again given by Equations (51) and (52) and those for multipath 2 given by

$$a_2 = 0.4 \exp(i\phi_2) \tag{53}$$

$$\frac{\tau_2}{T_C} = \eta_2 \tag{54}$$

where ϕ_2 is uniformly distributed $(0, 2\pi)$ and η_2 is uniformly distributed (0, 1). The multipath-induced bias error both with $(N=5,\ T = 0.5\ T_C)$ and without the cancellation filter are now shown in Figure 7. Again, each point is the average of 200 Monte Carlo realizations of $(a_1, \eta_1, a_2, \eta_2)$. Observe, that the filter again reduces the multipath-induced bias, but not by as much as for the case when only one multipath is present. Similar results are obtained when three multipaths are present. For example, we added a third multipath with $a_3 = 0.3 \exp(i\phi_3), \tau_3/T_C = \eta_3$ where ϕ_3 is randomly distributed $(0, 2\pi)$, η_3 is randomly distributed (0, 1). The results for this case are shown in Figure 8. Thus, in all situations explored, the filter was able to decrease the bias, but usually by only a factor of order 2.

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* Note that if the multipath delay is expected to exceed T_C then $N = 5$ taps, for $T = 0.5\ T_C$, is insufficient and one needs to add more taps.

It should be noted that the use of other tap delays T (i.e., $T = 0.25\ T_C$, $N = 9$ and $T = 0.35\ T_C$, $N = 7$) were explored, but none produced significantly better[*] performance than $T = 0.5\ T_C$.

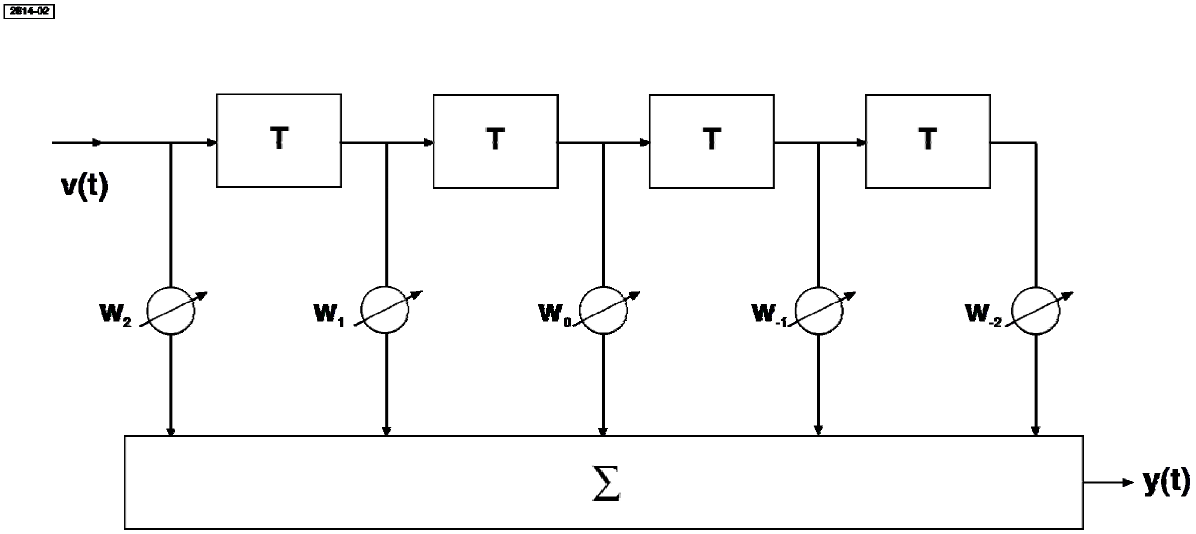

Figure 4. Five Tap FIR Filter

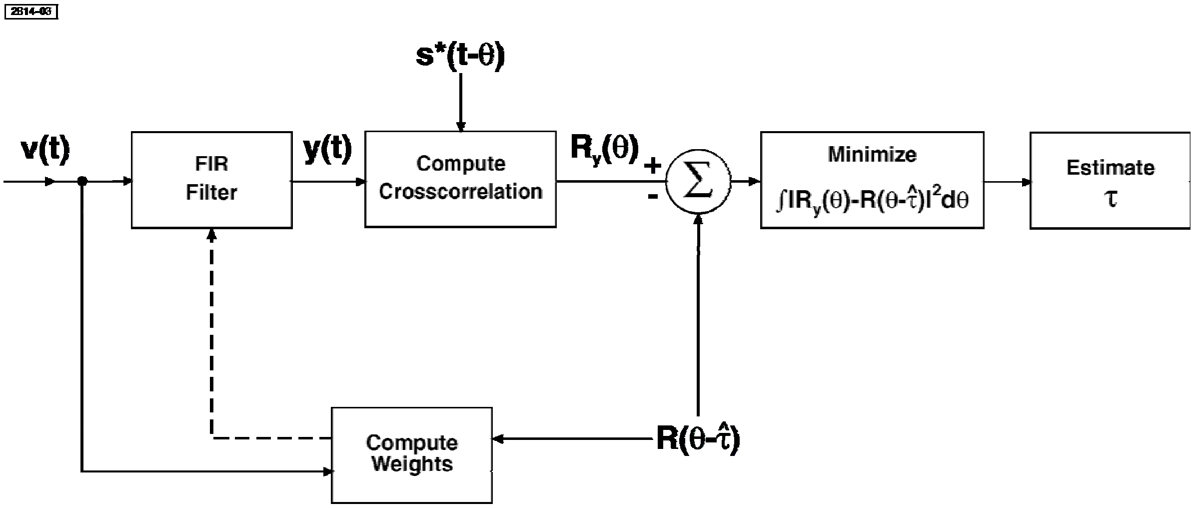

Figure 5. Block Diagram of Estimator

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[*] These smaller tap spacings produced better performance for small multipath delays, but worse performance for multipath delays of order T_C.

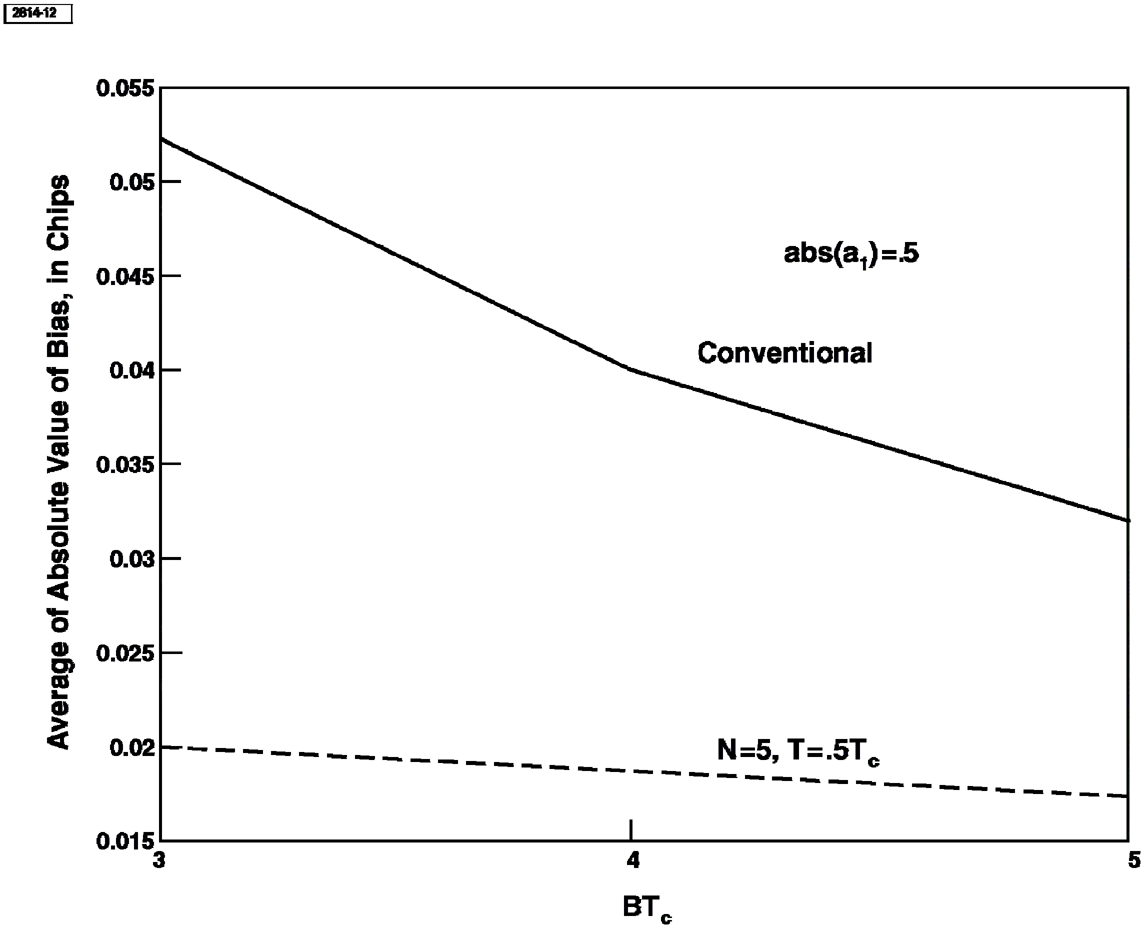

Figure 6. Comparison of Multipath Equalizer with Conventional for 1 Multipath

2814-18

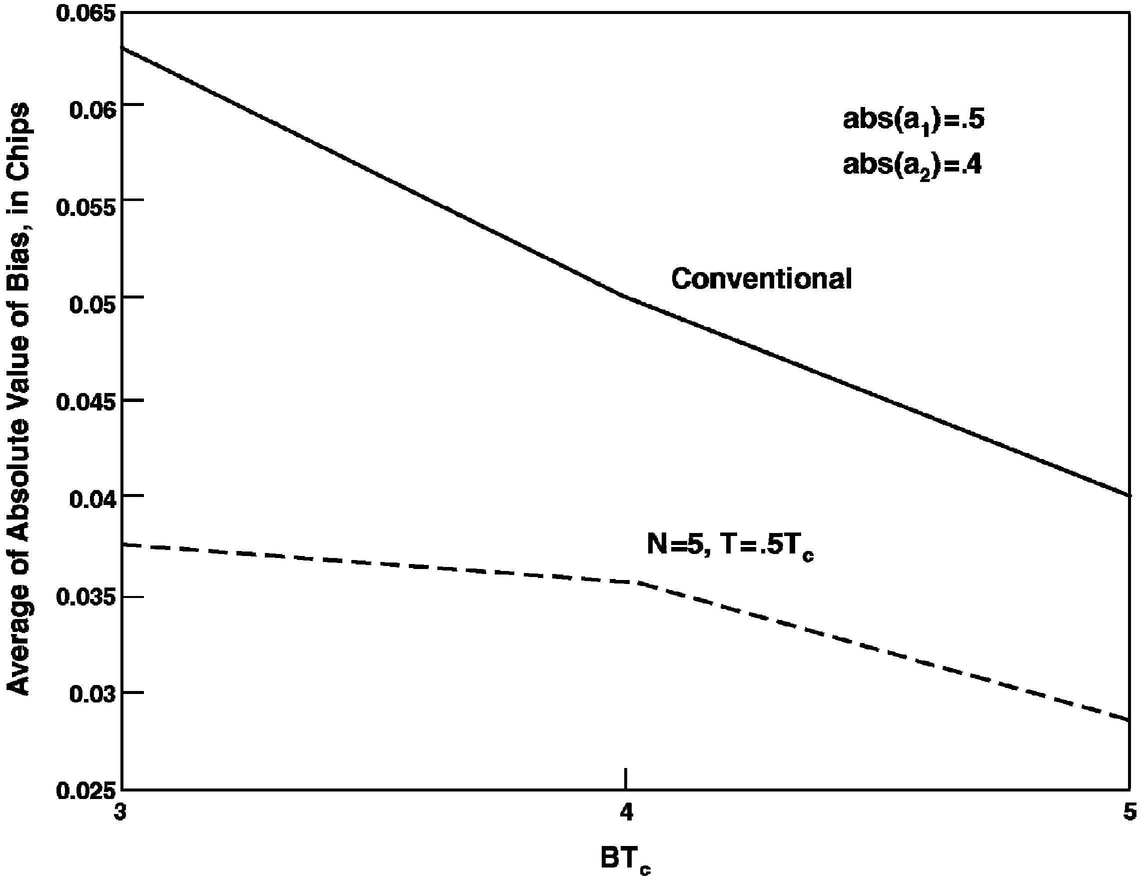

Figure 7. Comparison of Multipath Equalizer with Conventional for 2 Multipaths

2814-14

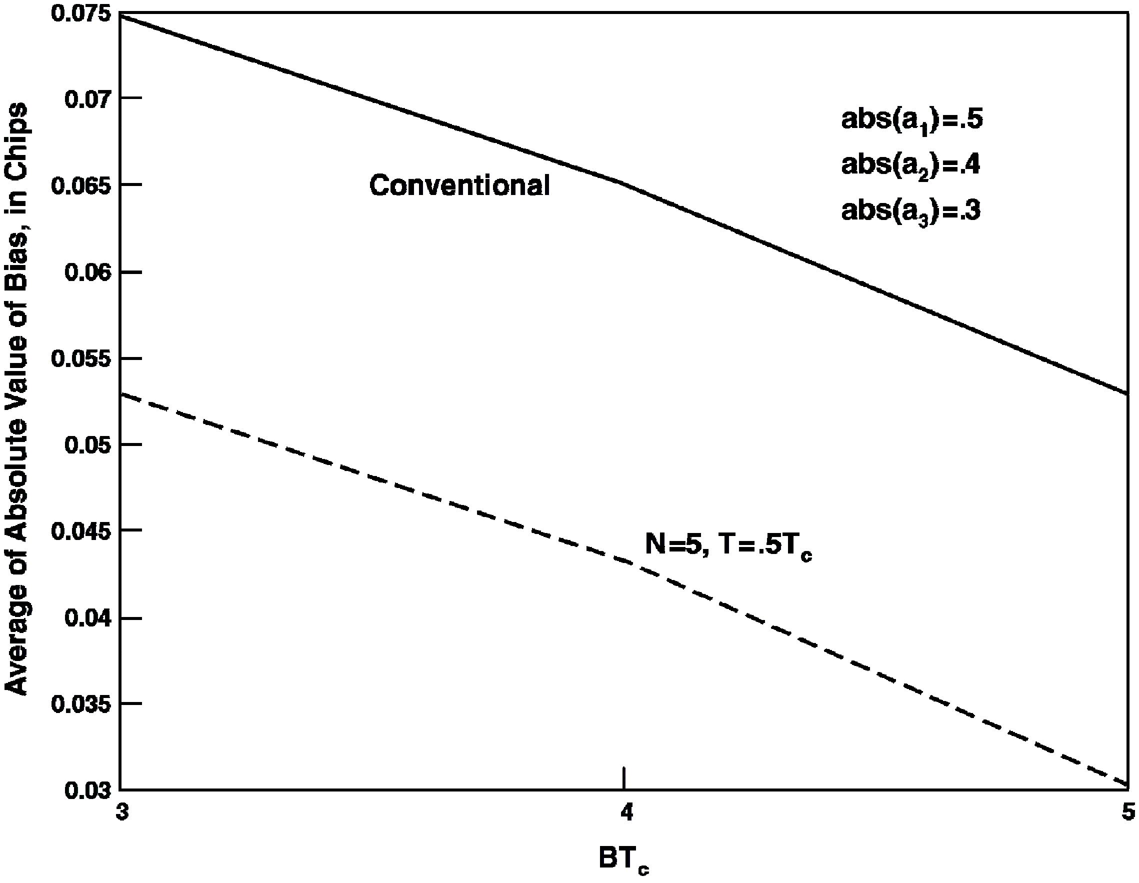

Figure 8. Comparison of Multipath Equalizer with Conventional for 3 Multipaths

Summary and Discussion

In the first portion of this paper we derived analytical expressions for the bias and variance introduced by multipath on a system that uses a noncoherent early-late gate discriminator to estimate time of arrival. These expressions were evaluated for the case of an early-late gate spacing δ much less than the chip duration T_C, and allowed us to quantitatively evaluate the advantage obtained by increasing the front-end bandwidth B when the signal is a PN sequence in the presence of white noise.

We then examined whether an adaptive multipath equalization filter could reduce the multipath-induced bias error further. We found that if an early-late gate discriminator is applied to find the peak of the generalized noncoherent, crosscorrelation function $Q_o(\hat{\tau})$ given in Equation (49), it is possible to reduce the multipath-induced bias by an additional factor of approximately two. The price paid for this bias reduction is the computation of the quantity

$$U_{km} = \int_{-T_C}^{T_C} d\theta \, r_v(\theta) r_v^*(\theta - |k-m|T)$$

for $|k-m| = 0,1 \ldots 2K$ and the function

$$S_k(\hat{\tau}) = \int_{-T_C}^{T_C} d\theta \, R(\theta) r_v(\theta + \hat{\tau} - kT)$$

for $k = -K \ldots 0, \ldots K$. For $N = 2K + 1 = 5$ this represents five computations of U_{km} and five computations of $S_k(\hat{\tau})$ for each $\hat{\tau}$.

We stress that all of the results presented are for the power spectrum given by Equation (10) and for signals with similar power spectra. Other signals, such as a binary offset carrier, which has a power spectrum that is very different from that in Equation (10), can be expected to yield very different results [22].

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Appendix

Bias Obtained Using Processor in Figure 5

In this Appendix we obtain an analytical expression for the bias when the peak in $Q_o(\hat{\tau})$ is estimated using an early-late gate discriminator. Let us write Q_o in terms of its scalar components. We get

$$Q_o = \sum_{n=-K}^{K} \sum_{m=-K}^{K} C_n^* \Gamma_{nm} C_m \tag{A1}$$

where Γ_{nm} is the *(n, m)* component of the inverse matrix Λ^{-1}.

We now compute component *n* of the vector *C*. First, substitute Equation (3) into (2) to get (for $\hat{\tau} = \theta + nT$)

$$\begin{aligned} r_v(\theta + nT) = \sum_{p=0}^{Q} a_p \frac{1}{T_o} \int_o^{T_o} dt\, s^*(t - \theta - nT) s(t - \tau - \tau_p) \\ + \frac{1}{T_o} \int_o^{T_o} dt\, s^*(t - \theta - nT) x(t) \end{aligned} \tag{A2}$$

where $a_o \equiv 1, \tau_o \equiv 0$ and the last term on the right-hand side of Equation (A2) is the noise contribution to the crosscorrelation. Now recall that the signal autocorrelation function is given by

$$R(\phi) = \frac{1}{T_o} \int_o^{T_o} dt\, s^*(t) s(t + \phi) \ . \tag{A3}$$

Consequently, by using Equation (A3) we can rewrite (A2) as

$$r_v(\theta + nT) = \sum_{p=0}^{Q} a_p R(\theta - \tau + T_{np}) + \rho_1(\theta + nT) \tag{A4}$$

where

$$T_{np} = nT - \tau_p \tag{A5}$$

$$\rho_1(\theta) = \frac{1}{T_o} \int_o^{T_o} dt\, s^*(t - \theta) x(t) \ . \tag{A6}$$

Therefore, component *n* of the vector *C* in Equation (45) is

$$\begin{aligned} C_n(\hat{\tau}) = \sum_{p=0}^{Q} \frac{a_p^*}{2T_1} \int_{-T_1}^{T_1} d\xi\, R(\xi + \hat{\tau} - \tau + T_{np}) R(\xi) \\ + \frac{1}{2T_1} \int_{-T_1}^{T_1} d\xi R(\xi) \rho_1^*(\xi + \hat{\tau} + nT) \end{aligned} \tag{A7}$$

where we have used the transformation $\xi = \theta - \hat{\tau}$ in Equation (45), along with the fact that $R(\xi)$ is a real function.

In order to calculate the bias in the estimate we now ignore the noise term (i.e., the second term on the right-hand side of Equation (A7)) and express $R(\xi)$ in terms of its power spectrum $P(f)$. For $T_1 \geq T_C$ we obtain*

$$C_n(\hat{\tau}) = C_S^2 \sum_{p=0}^{Q} a_p^* \int df \; P^2(f) \cos\left[2\pi f\left(\hat{\tau} - \tau + T_{np}\right)\right] \tag{A8}$$

where we have used the fact that $P(f)$ is a symmetric function of frequency f. As was the case in Section 2 of this paper, $P(f)$ is normalized such that along with the definitions its integral over all frequencies is unity, and C_S is the carrier signal power.

The other quantity in Equation (A1) that we must compute is the covariance matrix Λ, and its inverse Γ. If we use Equation (A3) in (46) it is readily seen that the term Λ_{nm} *is*

$$\begin{aligned}\Lambda_{nm} = \frac{1}{2T_1} \int_{-T_1}^{T_1} d\theta \left[\sum_{p=0}^{Q} a_p^* R^*\left(\theta - \tau + T_{np}\right) + \rho_1^*\left(\theta + nT\right) \right] \\ \bullet \left[\sum_{q=0}^{Q} a_q R\left(\theta - \tau + T_{mq}\right) + \rho_1\left(\theta + mT\right) \right] .\end{aligned} \tag{A9}$$

We now take an expectation of Equation (A9) and use the fact that $\langle \rho_1 \rangle = \langle \rho_1^* \rangle = 0$, along with the definitions

$$R(\theta) = \int_{-B/2}^{B/2} df P(f) e^{i2\pi f\theta} \tag{A10}$$

$$R_{xx}(\theta) = \int_{-B/2}^{B/2} df P_{xx}(f) e^{i2\pi f\theta} \tag{A11}$$

where $R_{xx}(\theta) = \langle x(t) x^*(t+\theta) \rangle$. We then find

$$\begin{aligned}\Lambda_{nm} = \frac{C_S^2}{2T_1} \sum_{p=0}^{Q} \sum_{q=0}^{Q} a_p^* a_q \int_{-B/2}^{B/2} df P^2(f) \cos 2\pi f\left(T_{np} - T_{mq}\right) \\ + \frac{C_S}{T_o} \int_{-B/2}^{B/2} df P_{xx}(f) P(f) \cos 2\pi f (n-m) T \; .\end{aligned} \tag{A12}$$

If we use Equation (10) *for* $P(f)$, assume the noise is white so that $P_{xx}(f) = N_o$, set $T_1 = T_C$ and define $\xi = 2f/B$ we obtain

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* For a PN signal $R(\xi) = 0$ for $|\xi| \geq T_C$, so that if $T_1 \geq T_C$ we can replace the integration limits in Equation (A7) by ∞.

$$\begin{aligned}\frac{\Lambda_{nm}}{C_S^2} &= \frac{1}{2}\sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p^* a_q \beta \int_0^1 d\xi \,\mathrm{sinc}^4\left(\frac{\pi}{2}\beta\xi\right)\cos\left[\pi\beta\left(\frac{T_{np}-T_{mq}}{T_C}\right)\xi\right] \\ &\quad + \frac{N_o}{C_S T_o}\beta\int_0^1 d\xi\,\mathrm{sinc}^2\left(\frac{\pi}{2}\beta\xi\right)\cos\left[\pi\beta(n-m)\frac{T\xi}{T_C}\right]\end{aligned} \tag{A13}$$

where $\beta = BT_C$. For sufficiently large integration times T_o the second term on the right-hand side of Equation (A13) will be small in comparison with the first. Once the components of Λ are known, its inverse Γ can be computed.

The bias error is now determined by substituting Equation (A8) into (49), using (49) in (50) and then setting $e(\hat{\tau}) = 0$. If in Equation (A8) we approximate $\cos 2\pi f\left(\varepsilon + T_{np}\right)$ by $\cos 2\pi f T_{np} - 2\pi f \varepsilon \sin 2\pi f T_{np}$, where $\varepsilon = \hat{\tau} - \tau$, we find

$$\hat{\tau} = \tau + \frac{b_1}{K_1} \tag{A14}$$

where

$$\begin{aligned} b_1 = -2C_S^2 &\sum_{n=-K}^{K}\sum_{m=-K}^{K}\sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p a_q^* \Gamma_{nm} \\ &\bullet\left[\overline{X}_{np}\overline{Z}_{mq} + \overline{Z}_{np}\overline{X}_{mq}\right]\end{aligned} \tag{A15}$$

$$\begin{aligned} K_1 = 4\pi C_S^2 &\sum_{n=-K}^{K}\sum_{m=-K}^{K}\sum_{p=0}^{Q}\sum_{q=0}^{Q} a_p a_q^* \Gamma_{nm} \\ &\bullet\left[\overline{X}_{np}\overline{Y}_{mq} + \overline{Y}_{np}\overline{X}_{mq} - \overline{Z}_{np}\overline{H}_{mq} - \overline{H}_{np}\overline{Z}_{mq}\right]\end{aligned} \tag{A16}$$

and $\overline{X}, \overline{Y}, \overline{Z}, \overline{H}$ are the same as X, Y, Z, H defined in Equations (14)-(17) except that $P(f)$ in (14)-(17) is replaced by $P^2(f)$ in $\overline{X}, \overline{Y}, \overline{Z}, \overline{H}$, and τ_p is replaced by $\tau_p - nT$ and τ_q is replaced by $\tau_q - mT$.

The term b_1/K_1 in Equation (A14) represents the bias on the estimate $\hat{\tau}$ of the true signal delay τ. It is readily shown that $b_1 = 0$ when the multipath is absent (i.e. $a_1 = a_2 \cdots = a_k = 0$), so that the estimate is unbiased in the absence of multipath.

Glossary

a_o = strength of direct path signal $\equiv 1$
a_p = complex strength of multipath p (p = 1, 2 …)
B = front-end bandwidth
C_S = signal carrier power
N = number of taps in FIR filter = $2K + 1$
N_o = white-noise power spectral density
$P(f)$ = signal power spectral density
$P_{xx}(f)$ = noise power spectral density
Q = total number of multipaths
S/N = input signal-to-noise ratio = $C_S/(N_o B)$
T = intertap delay in FIR filter
T_C = chip duration
T_o = integration time of crosscorrelator
δ = delay between early and late gates
τ = true direct-path delay
$\hat{\tau}$ = estimate of direct-path delay
τ_p = delay of multipath p relative to direct path
τ_o = 0 (by definition)