Capacity Improvement, Penalties, and the Fixed Charge Transportation Problem

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Received December 1996; revised May 1998; accepted 9 October 1998

Abstract: Capacity improvement and conditional penalties are two computational aides for fathoming subproblems in a branch-and-bound procedure. In this paper, we apply these techniques to the fixed charge transportation problem (*FCTP*) and show how relaxations of the *FCTP* subproblems can be posed as concave minimization problems (rather than LP relaxations). Using the concave relaxations, we propose a new conditional penalty and three new types of capacity improvement techniques for the *FCTP*. Based on computational experiments using a standard set of *FCTP* test problems, the new capacity improvement and penalty techniques are responsible for a three-fold reduction in the CPU time for the branch-and-bound algorithm and nearly a tenfold reduction in the number of subproblems that need to be evaluated in the branch-and-bound enumeration tree. © 1999 John Wiley & Sons, Inc. Naval Research Logistics 46: 341–355, 1999

1. INTRODUCTION

The fixed charge transportation problem (*FCTP*) involves the determination of a minimum cost method for transporting flow in a bipartite network in which, for each arc in the network, a quantum of cost must be incurred before any flow can be carried on that arc. Solution methods for *FCTP*'s fall into two general categories: approximate and exact. A review of approximate methods is contained in the survey paper by Guisewite and Pardalos [10]. Other approximate methods published subsequent to this survey paper are given in Diaby [7] and Kim and Pardalos [13]. The focus of the present paper, however, is on exact procedures. The vast majority of exact methods—including those proposed by Barr, Glover, and Klingman [2], Cabot and Erenguc [5], Gray [9], Kennington [11], Kennington and Unger [12], Lamar and Wallace [18], Palekar, Karwan, and Zionts [20], and Rech and Barton [21]—are based on implementations of a branch-and-bound algorithm for the *FCTP*.

Recent developments in branch-and-bound methods for *FCTP*'s have focused on conditional penalties (sometimes called "up and down" penalties) to obtain lower bounds to the *FCTP*. Barr, Glover, and Klingman [2] and Kennington and Unger [12] applied the Driebeek [8] penalties

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 CCC 0894-069X/99/040341-15

(referred to in this paper as "Type I" penalties) to the *FCTP*. For the *FCTP*, the conditional penalties proposed by Tomlin [26] are the same as the Type I penalties. Cabot and Erenguc [5] derived another type of conditional penalties (referred to in this paper as "Type II" penalties) for the *FCTP* that incorporate the fixed charges on selected arcs in the *FCTP*. Palekar, Karwan, and Zionts [20] and Lamar and Wallace [18] proposed conditional penalties (referred to in this paper as "Type III" penalties) for the *FCTP* which combine the Type I and Type II penalties.

The Type I penalties can be obtained using parametric analysis of the LP relaxation of the *FCTP*, whereas the Type II and Type III penalties can be interpreted as the parametric analysis on a *concave* relaxation of the *FCTP*. The idea of basing conditional penalties on a concave relaxation for fixed charge problems was originally suggested by Cabot and Erenguc [5, 6]. In this paper, we formalize the concept of a concave relaxation of the *FCTP*. In addition, we introduce new penalties (which we refer to as "Type IV" penalties) that are also based on a concave relaxation of the *FCTP*. The Type IV penalties are stronger than the Type I, Type II, and Type III penalties.

Another enhancement to the branch-and-bound procedure for the *FCTP* involves the use of "capacity improvement" (also called "range reduction" or "domain contraction"). Capacity improvement systematically "shrinks" the feasible region of the *FCTP* without "cutting off" the optimal solution to the *FCTP*. Capacity improvement techniques based on parametric analysis of an LP relaxation of the original problem have been developed by Lamar [14, 15], Lamar and Sheffi [16], Lamar, Sheffi, and Powell [17], Ryoo and Sahinidis [22], Sahinidis [23], and Thakur [25]. We refer to these methods as "Type I" capacity improvement. In this paper, we introduce three new capacity improvement techniques (referred to as "Type II," "Type III," and "Type IV") that are based on *concave* relaxations of the *FCTP*, rather than the LP relaxation. The concave relaxations generate tighter bounds in the capacity improvement procedure.

Although conditional penalties and capacity improvement are conceptually different, they are both based on the same type of parametric analysis of a (linear or concave) relaxation of the *FCTP*. Therefore, a certain amount of synergy can be achieved by combining these two techniques. In fact, based on computational experiments for a series of *FCTP* test problems, combining capacity improvement with conditional penalties using a concave relaxation of the *FCTP* causes nearly a 90% reduction in the average number of branch-and-bound subproblems that have to be evaluated and a two-thirds reduction in the average CPU time.

This paper is organized as follows. Section 2 formulates the *FCTP* and its LP relaxation. Section 3 describes several concave relaxations to the *FCTP*. Sections 4 and 5 present, respectively, the four types of conditional penalties and capacity improvement procedures considered in this paper. Section 6 compares the computational performance of these methods using a series of standard *FCTP* test problems. Finally, Section 7 summarizes the paper.

2. PROBLEM FORMULATION

The *FCTP* can be formulated as

Problem *FCTP*:

$$\text{minimize} \quad \sum_{(i,j)\in A} \phi_{ij}(x_{ij}) \tag{1}$$

subject to

$$\sum_{j\in J} x_{ij} = s_i \quad \forall\, i \in I, \qquad \sum_{i\in I} x_{ij} = d_j \quad \forall\, j \in J, \qquad x_{ij} \geq 0 \quad \forall\, (i, j) \in A. \tag{2}$$

where

$$\phi_{ij}(x_{ij}) = \begin{cases} 0 & \text{if } x_{ij} = 0, \\ f_{ij} + c_{ij}x_{ij} & \text{if } x_{ij} > 0 \end{cases} \tag{3}$$

and I is the set of origin nodes, J is the set of destination nodes, $A = \{(i, j) : i \in I, j \in J\}$ is the set of arcs, s_i is the supply at the ith origin node, d_j is the demand at the jth destination node, c_{ij} is the unit cost for arc (i, j), f_{ij} is the fixed charge for arc (i, j), and x_{ij} is the decision variable measuring the flow carried on arc (i, j). It is assumed that $\sum_{i \in I} s_i = \sum_{j \in J} d_j$. Let x_{ij}^* denote the optimal value of x_{ij} in problem *FCTP*. Also, for any problem $\bullet$, let $v[\bullet]$ denote the optimal objective function value of problem $\bullet$. Thus, $v[FCTP] = \sum_{(i,j) \in A} \phi_{ij}(x_{ij}^*)$.

Typically, in the branch-and-bound procedure used to solve problem *FCTP*, a series of "subproblems" are formed by partitioning the arc set A into $A = A^{\mathrm{OUT}} \cup A^{\mathrm{IN}} \cup A^{\mathrm{FREE}}$. For each arc $(i, j) \in A^{\mathrm{OUT}}$ we have $x_{ij} = 0$; for each arc $(i, j) \in A^{\mathrm{IN}}$ we have $x_{ij} > 0$; and for each arc $(i, j) \in A^{\mathrm{FREE}}$ we have $x_{ij} \geq 0$. In the branch-and-bound procedure, let P denote the subproblem of *FCTP* currently under consideration. Note that any subproblem P can be expressed in the form of the original problem *FCTP* by appropriately assigning values to the fixed charges f_{ij}. Specifically, for each arc $(i, j) \in A^{\mathrm{OUT}}$, f_{ij} is replaced with a large positive constant in subproblem P. Similarly, for each arc $(i, j) \in A^{\mathrm{IN}}$, the original value of f_{ij} is accumulated as a "sunk" investment and f_{ij} is replaced with zero in subproblem P.

Let $\bar{P}$ denote the LP relaxation of subproblem P. As observed by Balinski [1], problem $\bar{P}$ can be specified in the form of the following (linear) transportation problem:

Problem $\bar{P}$:

$$\text{minimize} \qquad \sum_{(i,j) \in A} (c_{ij} + f_{ij}/u_{ij})x_{ij} \tag{4}$$

subject to (2),

where

$$u_{ij} = \min\{s_i, d_j\}. \tag{5}$$

Referring to the LP relaxation $\bar{P}$, let $\bar{x}_{ij}$ denote the optimal value of x_{ij}, let $\bar{a}_{ijkl}$ denote the element in the (i, j)th column and the (k, l)th row [i.e., (k, l)th basic variable] in the optimal simplex tableau, let $\bar{\pi}_i$ and $\bar{\sigma}_j$ denote, respectively, the optimal value of the dual variables associated with the ith origin constraint and the jth destination constraint [see Eq. (2)], let $\bar{r}_{ij} = (c_{ij} + f_{ij}/u_{ij}) - \bar{\pi}_i - \bar{\sigma}_j$ denote the reduced cost for arc (i, j), and let $\bar{z}_{ij} = \bar{\pi}_i + \bar{\sigma}_j$. In addition, we partition the arc set A into $A = \bar{B} \cup \bar{N}$ such that $(i, j) \in \bar{B}$ if x_{ij} is basic in the optimal solution to $\bar{P}$ and $(i, j) \in \bar{N}$ if x_{ij} is nonbasic in the optimal solution to $\bar{P}$. For each arc $(k, l) \in \bar{B}$, the following two arc subsets are also defined:

$$\bar{N}_{kl}^{\mathrm{UP}} = \{(i, j) : (i, j) \in \bar{N} \qquad \text{and} \qquad \bar{a}_{ijkl} < 0\},$$

$$\bar{N}_{kl}^{\mathrm{DN}} = \{(i, j) : (i, j) \in \bar{N} \qquad \text{and} \qquad \bar{a}_{ijkl} > 0\}. \tag{6}$$

Note that, for the *FCTP*, all $\bar{a}_{ijkl}$ are ± 1 or 0.

The information given above describing the optimal solution to the LP relaxation $\bar{P}$ is used to define several *concave* relaxations to problem P.

3. CONCAVE RELAXATIONS

Let Q denote a concave relaxation of problem P. We consider four different types of concave relaxations denoted by the superscript t, where t = I, II, III, or IV. We first formulate the Type I, II, and III relaxations and describe several important properties pertaining to the parametric analysis of these relaxations. We then extend our analysis to the Type IV relaxation.

3.1. Type I, II, and III

For t = I, II, and III, the concave relaxation Q^t is formulated as follows:
Problem Q^t:

$$\text{minimize} \qquad \sum_{(i,j)\in A} \psi_{ij}^t(x_{ij}) \tag{7}$$

subject to

$$\sum_{j\in J} x_{ij} = s_i \qquad \forall\, i \in I, \qquad \sum_{i\in I} x_{ij} = d_j \qquad \forall\, j \in J, \tag{8}$$

$$x_{ij} \qquad \text{unconstrained} \qquad \text{if } (i, j) \in \bar{B}, \tag{9}$$

$$x_{ij} \geq 0 \qquad \text{if } (i, j) \in \bar{N}. \tag{10}$$

For Type t = I, the objective function for each arc $(i, j) \in A$ is specified as the linear function

$$\psi_{ij}^{\mathrm{I}}(x_{ij}) = (c_{ij} + f_{ij}/u_{ij})x_{ij}, \tag{11}$$

where u_{ij} is defined in Eq. (5).

For Type t = II, the objective function for each arc $(i, j) \in A$ is specified as the fixed charge function

$$\psi_{ij}^{\mathrm{II}}(x_{ij}) = \begin{cases} (c_{ij} + f_{ij}/u_{ij})x_{ij} & \text{if } (i, j) \in \bar{B}, \\ 0 & \text{if } (i, j) \in \bar{N} \text{ and } x_{ij} = 0, \\ f_{ij} + c_{ij}x_{ij} & \text{if } (i, j) \in \bar{N} \text{ and } x_{ij} > 0 \text{ and } c_{ij} \geq \bar{z}_{ij}, \\ \bar{r}_{ij}u_{ij} + \bar{z}_{ij}x_{ij} & \text{if } (i, j) \in \bar{N} \text{ and } x_{ij} > 0 \text{ and } c_{ij} < \bar{z}_{ij}, \end{cases} \tag{12}$$

where $\bar{r}_{ij}$ is the reduced cost for arc (i, j) in the LP problem $\bar{P}$ and $\bar{z}_{ij} = \bar{\pi}_i + \bar{\sigma}_j$. The reason for choosing a fixed charge function of this form is discussed shortly [following Eq. (14)].

For Type t = III, the objective function for each arc $(i, j) \in A$ is specified using either Eq. (11) or Eq. (12), depending upon which function produces the largest value in the parametric analysis of problem Q^{III}. Thus, we specify $\psi_{ij}^{\mathrm{III}}(x_{ij})$ as

$$\psi_{ij}^{\mathrm{III}}(x_{ij}) = \max\{\psi_{ij}^{\mathrm{I}}(x_{ij}),\ \psi_{ij}^{\mathrm{II}}(x_{ij})\}. \tag{13}$$

Note that, although $\psi_{ij}^{\mathrm{III}}(x_{ij})$ is not a concave function, we can interpret Q^{III} as a concave minimization problem by selecting the appropriate concave function $[\psi_{ij}^{\mathrm{I}}(x_{ij})$ or $\psi_{ij}^{\mathrm{II}}(x_{ij})]$ for each arc $(i, j) \in A$. This idea was originally proposed by Palkar, Karwan, and Zionts [20].

In Q^t for $t = \mathrm{I}$, II, and III, we also define a reduced cost *function*, denoted $\Delta_{ij}^t(x_{ij})$, for each arc $(i, j) \in \bar{N}$ as follows:

$$\Delta_{ij}^t(x_{ij}) = \psi_{ij}^t(x_{ij}) - \bar{z}_{ij}x_{ij}. \tag{14}$$

The reduced cost function in Eq. (14) is a generalization of the reduced cost *coefficient* $\bar{r}_{ij}$ for the LP relaxation. In fact, for $t = \mathrm{I}$, $\Delta_{ij}^{\mathrm{I}}(x_{ij}) = \bar{r}_{ij}x_{ij}$. Note that, for $t = \mathrm{I}$, II, and III, $\psi_{ij}^t(x_{ij})$ has been chosen so that $\Delta_{ij}^t(x_{ij})$ will be a nondecreasing, concave function for $x_{ij} \geq 0$ and that $\Delta_{ij}^t(0) = 0$. This is why $\psi_{ij}^{\mathrm{II}}(x_{ij})$ is set to $\bar{r}_{ij}u_{ij} + \bar{z}_{ij}x_{ij}$ if $(i, j) \in \bar{N}$ and $x_{ij} > 0$ and $c_{ij} < \bar{z}_{ij}$ in Eq. (12).

We also let "$Q^t \mid x_{ij} = \bar{x}_{ij} + \delta$" denote problem Q^t augmented with the constraint "$x_{ij} = \bar{x}_{ij} + \delta$" where $\bar{x}_{ij}$ is the optimal value of x_{ij} in the LP relaxation $\bar{P}$ and δ is a (as yet unspecified) scalar. Much of our subsequent analysis involves the change in the optimal objective function value of problem "$Q^t \mid x_{ij} = \bar{x}_{ij} + \delta$" as δ is varied. Hence, we define a function, denoted $\theta_{ij}^t(\delta)$, as follows:

$$\theta_{ij}^t(\delta) = v[Q^t \mid x_{ij} = \bar{x}_{ij} + \delta] - v[Q^t]. \tag{15}$$

We refer to $\theta_{ij}^t(\delta)$ as the "parametric function" for problem Q^t and arc (i, j).

Problem Q^t, for $t = \mathrm{I}$, II, and III, exhibits four special properties that makes $\theta_{ij}^t(\delta)$ easy to evaluate. First, observe that, by construction, Q^t is a relaxation of problem P. Second, also by construction, the optimal solution to the LP relaxation $\bar{P}$ is also an optimal solution to problem Q^t and $v[Q^t] = v[\bar{P}]$. To see this, note that $\{\bar{x}_{ij}\}$ is feasible in Q^t and that $\Delta_{ij}^t(x_{ij})$ is a nondecreasing function for each $(i, j) \in \bar{N}$. Third, if $(i, j) \in \bar{N}$, then $\theta_{ij}^t(\delta)$ is given by the reduced cost function for arc (i, j). That is, if $(i, j) \in \bar{N}$, then

$$\theta_{ij}^t(\delta) = \begin{cases} \Delta_{ij}^t(\delta) & \text{if } \delta \geq 0, \\ +\infty & \text{if } \delta < 0. \end{cases} \tag{16}$$

Fourth, if $(k, l) \in \bar{B}$, then forcing x_{kl} to shift from $\bar{x}_{kl}$ to $\bar{x}_{kl} + \delta$ in problem Q^t means that a *single* nonbasic variable x_{ij} will change from zero to $|\delta|$ and the minimum objective function value will increase by $\Delta_{ij}^t(|\delta|)$. The reason a single nonbasic arc will change is that the basic arcs are not restricted in problem Q^t [see Eq. (9)] and the reduced cost functions $\Delta_{ij}^t(x_{ij})$ are nondecreasing and concave for the nonbasic arcs. Thus, if $(k, l) \in \bar{B}$, then the function $\theta_{kl}^t(\delta)$ is given by

$$\theta_{kl}^t(\delta) = \left\{ \begin{array}{ll} 0 & \text{if } \delta = 0 \\ \min\limits_{(i,j)\in \bar{R}_{kl}^{\mathrm{DN}}} \{\Delta_{ij}^t(-\delta)\} & \text{if } \delta < 0 \\ \min\limits_{(i,j)\in \bar{R}_{kl}^{\mathrm{UP}}} \{\Delta_{ij}^t(\delta)\} & \text{if } \delta > 0 \end{array} \right\} \tag{17}$$

where $\bar{N}_{kl}^{\mathrm{DN}}$ and $\bar{N}_{kl}^{\mathrm{UP}}$ are specified in Eq. (6). Observe that for t = I, II, and III and for each $(i, j) \in A$, $\theta_{ij}^{t}(\delta)$ is a nonnegative, unimodal function of δ and that $\theta_{ij}^{t}(0) = 0$.

This type of parametric analysis can also be extended to the Type IV concave relaxation.

3.2. Type IV

For Type t = IV, we consider a *family* of concave relaxations denoted by Q_{mn}^{IV}, where $(m, n) \in \bar{N}$. Problem Q_{mn}^{IV} is of the same form as problem Q^{III} *plus* the additional capacity constraint

$$x_{mn} \le u_{mn} \tag{18}$$

for a nonbasic arc (m, n). Thus, for problem Q_{mn}^{IV}, the objective function is given by $\psi_{ij}^{\mathrm{IV}}(x_{ij}) = \psi_{ij}^{\mathrm{III}}(x_{ij})$ [see Eq. (13)] and the feasible region is defined by Eqs. (8), (9), (10), and (18). For a given arc $(i, j) \in A$ and a given value of δ, the arc (m, n) in problem Q_{mn}^{IV} is selected as the nonbasic arc whose flow is shifted from zero to $|\delta|$ in response to the constraint "$x_{ij} = \bar{x}_{ij} + \delta$" being added to problem Q^{III}. In other words,

$$(m, n) = \arg\operatorname{index}\{\theta_{ij}^{\mathrm{III}}(\delta)\}, \tag{19}$$

where $\theta_{ij}^{\mathrm{III}}(\delta)$ is given by Eqs. (16) and (17) for t = III. If $(i, j) \in \bar{N}$, then arc (m, n) is the same as arc (i, j). If arc $(i, j) \in \bar{B}$, then (m, n) will be a single arc contained in the set $\bar{N}_{ij}^{\mathrm{DN}}$ for $\delta < 0$, and a single arc contained in the set $\bar{N}_{ij}^{\mathrm{UP}}$ for $\delta > 0$.

In evaluating $\theta_{ij}^{\mathrm{IV}}(\delta)$, we consider two possible cases: $|\delta| \le u_{mn}$ and $|\delta| > u_{mn}$. For the first case, if $|\delta| \le u_{mn}$, then constraint (18) is not binding and the parametric analysis of problem Q_{mn}^{IV} will be identical to that of problem Q^{III}. Thus, if $|\delta| \le u_{mn}$, then $\theta_{ij}^{\mathrm{IV}}(\delta) = \theta_{ij}^{\mathrm{III}}(\delta)$. For the second case, if $|\delta| > u_{mn}$, then either (i) there will zero units of flow shifted onto arc (m, n) and $|\delta|$ units shifted onto another nonbasic arc or (ii) there will be u_{mn} units shifted to arc (m, n) and $|\delta| - u_{mn}$ units shifted onto another nonbasic arc.

To describe this process algebraically, we define $\hat{\theta}_{ij}^{\mathrm{III}}(\delta)$ as the parametric function for problem Q^{III} with nonbasic arc (m, n) determined according to Eq. (19) *excluded*. That is, $\hat{\theta}_{ij}^{\mathrm{III}}(\delta)$ is defined by Eqs. (16) and (17) except that the set of nonbasic arcs is taken as $\bar{N} - \{(m, n)\}$ in these equations. The function $\hat{\theta}_{ij}^{\mathrm{III}}(\delta)$ can be interpreted as the "second best" value for $\theta_{ij}^{\mathrm{III}}(\delta)$. Then, $\theta_{ij}^{\mathrm{IV}}(\delta)$, the parametric function for problem Q_{mn}^{IV}, is given by

$$\theta_{ij}^{\mathrm{IV}}(\delta) = \begin{cases} \theta_{ij}^{\mathrm{III}}(\delta) & \text{if } |\delta| \le u_{mn} \\ \min\{\hat{\theta}_{ij}^{\mathrm{III}}(\delta),\ \Delta_{mn}^{\mathrm{III}}(u_{mn}) + \hat{\theta}_{ij}^{\mathrm{III}}(\tilde{\delta})\} & \text{if } |\delta| > u_{mn} \end{cases} \tag{20}$$

where

$$\tilde{\delta} = \begin{cases} \delta + u_{mn} & \text{if } \delta < 0, \\ \delta - u_{mn} & \text{if } \delta > 0. \end{cases} \tag{21}$$

Observe that, for t = IV for each $(i, j) \in A$, $\theta_{ij}^{\mathrm{IV}}(\delta)$ is also a nonnegative, unimodal function of δ and that $\theta_{ij}^{\mathrm{IV}}(0) = 0$. Also, note that for any arc (m, n), problem Q^{III} is a relaxation of problem Q_{mn}^{IV}; and both problems Q^{I} and Q^{II} are relaxations of problem Q^{III}. Hence, for any arc $(i, j) \in A$ and any value of δ, the following relationships hold:

$$\theta_{ij}^{\mathrm{IV}}(\delta) \geq \theta_{ij}^{\mathrm{III}}(\delta) \geq \theta_{ij}^{\mathrm{II}}(\delta) \qquad \text{and} \qquad \theta_{ij}^{\mathrm{IV}}(\delta) \geq \theta_{ij}^{\mathrm{III}}(\delta) \geq \theta_{ij}^{\mathrm{I}}(\delta) \tag{22}$$

The parametric function $\theta_{ij}^t(\delta)$ given in Eqs. (16), (17), and (20) is used to compute the conditional penalties and the capacity improvement coefficients for subproblem P.

4. CONDITIONAL PENALTIES

Conditional penalties are used to obtain a lower bound to $v[P]$ that is tighter than the LP relaxation value $v[\bar{P}]$. In this section, we consider four types of conditional penalties, denoted PEN^t for t = I, II, III, and IV, that are associated with the four types of concave relaxations formulated in Section 3.

Let $(k, l) \in \bar{B} \cap A^{\mathrm{FREE}}$ and let DN_{kl}^t be an underestimate of the increase in the objective function value of P when arc (k, l) is transferred from A^{FREE} to A^{OUT}. Since Q^t for t = I, II, and III and Q_{mn}^{IV} for each $(m, n) \in \bar{N}$ are relaxations of problem P, the parametric function $\theta_{kl}^t(\delta)$ can be used to obtain DN_{kl}^t. Specifically,

$$DN_{kl}^t = \theta_{kl}^t(-\bar{x}_{kl}). \tag{23}$$

In addition, let UP_{kl}^t be an underestimate of the increase in the objective function value of P when arc (k, l) is transferred from A^{FREE} to A^{IN}. In this case, the objective function will increase either by (i) the remainder of the fixed charge f_{kl} or (ii) an amount given by $\theta_{kl}^t(\delta)$. Specifically,

$$UP_{kl}^t = \min\{f_{kl} \cdot (1 - \bar{x}_{kl}/u_{kl}),\ \theta_{kl}^t(u_{kl} - \bar{x}_{kl})\}. \tag{24}$$

The type t conditional penalty for arc (k, l), denoted PEN_{kl}^t, is given by

$$PEN_{kl}^t = \min\{DN_{kl}^t, UP_{kl}^t\} \tag{25}$$

This penalty is computed for each arc $(k, l) \in \bar{B} \cap A^{\mathrm{FREE}}$ and the maximum penalty is taken as the type t conditional penalty. That is,

$$PEN^t = \max_{\substack{(k,l)\in\\ \bar{B}\cap A^{\mathrm{FREE}}}} \{PEN_{kl}^t\}. \tag{26}$$

This penalty is used in the following "fathoming" criterion for subproblem P:

$$v[\bar{P}] + PEN^t \geq ub[FCTP] \tag{27}$$

Here, $ub[FCTP]$ is an upper bound to $v[FCTP]$. Typically, $ub[FCTP]$ is taken as the objective function value of the current incumbent solution to $FCTP$ in the branch-and-bound procedure. If condition (27) is satisfied, then subproblem P (and its "descendants") can be excluded from further consideration in the branch-and-bound procedure.

Because of the conditions given in (22), the following relationships hold among the conditional penalties:

$$PEN^{\mathrm{IV}} \geq PEN^{\mathrm{III}} \geq PEN^{\mathrm{II}} \geq 0 \qquad \text{and} \qquad PEN^{\mathrm{IV}} \geq PEN^{\mathrm{III}} \geq PEN^{\mathrm{I}} \geq 0. \tag{28}$$

The Type I conditional penalty, PEN^{I}, is the Driebeek–Tomlin penalty [8, 26]. This penalty can also be interpreted as the increase in the objective function value of $v[\bar{P}]$ obtained from a single dual simplex pivot. The Type II conditional penalty, PEN^{II}, is the Cabot and Erenguc [5] penalty. The Type II penalty can also be derived using a Lagrangian relaxation of problem P. The Type III conditional penalty, PEN^{III}, was originally proposed by Palekar, Karwan, and Zionts [20]. It was revised sightly by Lamar and Wallace [18] to ensure that PEN^{III} is a valid conditional penalty for problem P. The Type IV conditional penalty, PEN^{IV}, is a new conditional penalty for the *FCTP* being introduced in this paper. As indicated by the relationships in (28), it is a stronger penalty than the Type I, II, and III conditional penalties.

The parametric function $\theta_{ij}^t(\delta)$ used to compute PEN^t can also be applied to a capacity improvement procedure for the *FCTP*.

5. CAPACITY IMPROVEMENT

The purpose of capacity improvement is to obtain an "improved capacity coefficient," denoted $\hat{u}_{ij}^t$, for x_{ij} such that

$$\hat{u}_{ij}^t \le u_{ij}, \tag{29}$$

where u_{ij} is given by Eq. (5). Furthermore, if the optimal solution to problem *FCTP* is contained in the feasible region to subproblem P (i.e., if $v[FCTP] = v[P]$), then $\hat{u}_{ij}^t$ is chosen such that the following condition is also true:

$$x_{ij}^* \le \hat{u}_{ij}^t. \tag{30}$$

Here, x_{ij}^* is the optimal value of x_{ij} in problem *FCTP*. [A similar procedure can also be used to obtain an improved lower bound to x_{ij} that is greater than or equal to zero.] Let $\hat{P}^t$ denote the LP problem of the form of problem $\bar{P}$ except that $\hat{u}_{ij}^t$ is used in place of u_{ij} for all $(i, j) \in A^{\mathrm{FREE}}$. Note that, for the *FCTP*, problem $\hat{P}^t$ is also of the form of a (linear) transportation problem. Condition (29) means that $v[\hat{P}^t] \ge v[\bar{P}]$; and condition (30) implies that $v[\hat{P}^t] \le v[FCTP]$ whenever the optimal solution to problem *FCTP* is contained in subproblem P. Furthermore, the conditional penalties described in Section 4 can be computed using problem $\hat{P}^t$ (rather than problem $\bar{P}$) simply by solving $\hat{P}^t$ and using the coefficients $\hat{u}_{ij}^t$ (rather than u_{ij}). Therefore, the capacity improvement procedure produces the following "fathoming" criterion for subproblem P:

$$v[\hat{P}^t] + PEN^t \ge ub[FCTP]. \tag{31}$$

Criterion (31) is at least as tight as criterion (27). In addition, the improved capacity coefficients $\hat{u}_{ij}^t$ can also be used in place of u_{ij} in any "descendants" of subproblem P in the branch-and-bound procedure. This, in turn, facilitates the fathoming of these descendant subproblems. See Lamar [14] for details.

To compute $\hat{u}_{ij}^t$ that satisfies condition (30), we define a scalar, denoted $\hat{\delta}_{ij}^t$, as follows:

$$\hat{\delta}_{ij}^t = \max\{\delta : \theta_{ij}^t(\delta) \le ub[FCTP] - v[\bar{P}]\} \tag{32}$$

Remember that, by construction, $\theta_{ij}^t(\delta)$ is a nonnegative, unimodal function of δ with $\theta_{ij}^t(0) = 0$. Therefore, if $v[FCTP] = v[P]$, then $v[FCTP|x_{ij} > \bar{x}_{ij} + \hat{\delta}_{ij}^t] > ub[FCTP]$. Hence, if $v[FCTP] = v[P]$, then it must be the case that $x_{ij}^* \leq \bar{x}_{ij} + \lfloor \hat{\delta}_{ij}^t \rfloor$, where $\lfloor \hat{\delta}_{ij}^t \rfloor$ indicates the "floor" function (i.e., the greatest integer less than or equal to $\hat{\delta}_{ij}^t$). To ensure that the improved capacity coefficient also satisfies condition (29), we set $\hat{u}_{ij}^t$ to

$$\hat{u}_{ij}^t = \min\{u_{ij}, \bar{x}_{ij} + \lfloor \hat{\delta}_{ij}^t \rfloor\}. \tag{33}$$

Equation (33) is used to determine $\hat{u}_{ij}^t$ for each $(i, j) \in A^{\mathrm{FREE}}$. Problem $\hat{P}^t$ is then solved and used to obtain $v[\hat{P}^t]$ and PEN^t. These values, in turn, are used in criterion (31) in an attempt to fathom subproblem P. If criterion (31) is not satisfied, then either (i) the capacity improvement procedure can be repeated (using $\hat{P}^t$ in place of $\bar{P}$) to obtain even tighter improved capacity coefficients for the current subproblem P or (ii) an arc $(k, l) \in \bar{B} \cap A^{\mathrm{FREE}}$ can be selected as the "branching" arc and the improved capacity coefficients can be used in the descendants to the current subproblem.

Because of the conditions given in (22), the following relationships hold among the improved capacity coefficients:

$$\hat{u}_{ij}^{\mathrm{IV}} \leq \hat{u}_{ij}^{\mathrm{III}} \leq \hat{u}_{ij}^{\mathrm{II}} \leq u_{ij} \qquad \text{and} \qquad \hat{u}_{ij}^{\mathrm{IV}} \leq \hat{u}_{ij}^{\mathrm{III}} \leq \hat{u}_{ij}^{\mathrm{I}} \leq u_{ij}. \tag{34}$$

The type I improved capacity coefficients, $\hat{u}_{ij}^{\mathrm{I}}$, based on a linear relaxation of problem P, have been studied by Lamar [14, 15], Lamar and Sheffi [16], Lamar, Sheffi, and Powell [17], Ryoo and Sahinidis [22], Sahinidis [23], and Thakur [25]. The Type II, III, and IV capacity improvement coefficients $\hat{u}_{ij}^{\mathrm{II}}$, $\hat{u}_{ij}^{\mathrm{III}}$, and $\hat{u}_{ij}^{\mathrm{IV}}$ are new for the $FCTP$ and are being introduced in this paper.

The computational efficiencies achieved by the four different types of conditional penalties and capacity improvement are examined next.

6. COMPUTATIONAL PERFORMANCE

The purpose of the computational experiments described in this section is to compare the four different types of conditional penalties and capacity improvement techniques empirically. The experimental plan and the results are described below.

6.1. Test Problems and Procedure

The $FCTP$ test problems used in our experiments were based on those developed by Kennington and Unger [12] and Palekar, Karwan, and Zionts [20]. The characteristics of these test problems are summarized in Table 1. Three test problems for each of the 16 sets were generated, resulting in a total of 48 test problems. All test problems were fully dense (i.e., there was an arc from each origin node to each destination node). Except for the last demand node and the arcs incident to that node, all supplies, demands, fixed charges, and variable costs for each test problem were generated randomly from a uniform distribution with prespecified lower and upper limits as shown in Table 1. The demand at the last demand node was set so that total supply equalled total demand in the network; and all arcs incident to the last demand node had zero fixed charges and zero variable costs.

Table 1. Characteristics of test problems.

Set	Size $\lvert I\rvert \times \lvert J\rvert$	Range of c_{ij}	Range of f_{ij}	Range of s_i	Range of d_j
1	6×9	2–12	100–200	10–100	5–60
2	6×9	2–12	200–400	10–100	5–60
3	6×9	2–12	400–800	10–100	5–60
4	6×9	2–12	2000–4000	10–100	5–60
5	6×9	0–0	100–200	10–100	5–60
6	6×9	100–200	100–200	10–100	5–60
7	6×9	100–200	2–12	10–100	5–60
8	7×11	2–12	20–40	20–100	10–60
9	7×11	2–12	40–80	20–100	10–60
10	7×11	2–12	200–400	20–100	10–60
11	7×11	20–40	20–40	20–100	10–60
12	7×11	20–40	2–12	20–100	10–60
13	5×20	1–5	100–200	60–100	10–25
14	5×20	1–5	40–60	60–100	10–25
15	5×20	100–200	100–200	60–100	10–25
16	5×20	100–200	10–20	60–100	10–25

A branch-and-bound algorithm for the *FCTP* was programmed in the C language and was compiled using the Watcom C/C++ compiler version 10.0 and run on a 486DX2-66 personal computer. The LP relaxations of the subproblems (i.e., the linear transportation problems $\bar{P}$) were solved using the CPLEX callable library, version 3.0. Our branch-and-bound procedure allowed for five possible choices for conditional penalties (Type I, II, III, IV, and no conditional penalties) as well as five possible choices for capacity improvement (Type I, II, III, IV, and no capacity improvement). Thus, there were 25 possible combinations. Each of the 48 test problems was solved for each of the 25 possible combinations of conditional penalties and capacity improvement.

The branch-and-bound enumeration tree was evaluated using a depth-first search with backtracking. The lowest cost feasible solution using the *FCTP* objective function $\phi_{ij}(x_{ij})$ [see Eq. (3)] was retained as the current incumbent and its objective function value was stored as $ub[FCTP]$. If $v[\bar{P}]$ was greater than or equal to $ub[FCTP]$, then the current subproblem was "fathomed." If the current subproblem could not be fathomed by $v[\bar{P}]$ and conditional penalties were used, then the Type t conditional penalty, PEN^t, was calculated and added to $v[\bar{P}]$ in an effort to fathom the current subproblem. If the current subproblem still could not be fathomed and capacity improvement was used, then the Type t improved capacity coefficients $\bar{u}^t_{ij}$ were calculated. Experimentation indicated that it was more efficient to use these improved capacity coefficients on descendants of the current subproblem rather than to resolve the relaxation of the current subproblem. However, if conditional penalties were used and the "down" or "up" penalty for an arc indicated that that arc could be "pegged" (i.e., transferred from A^{FREE} to A^{OUT} or A^{IN}), then it was more efficient to resolve the relaxation of the current subproblem. If the current subproblem still could not be fathomed and conditional penalties were used, then the arc in A^{FREE} associated with the largest conditional penalty was selected as the "branching" arc and two new subproblems were formed. Otherwise, if conditional penalties were not used, then the arc in A^{FREE} whose value $\bar{x}_{ij}/u_{ij}$ was closest to 0.5 was selected as the branching arc.

Note that the branch-and-bound algorithm without conditional penalties or capacity improvement provided a benchmark for comparing the various types of conditional penalties and capacity improvement. In addition, each of the test problems was also solved separately using

Table 2. Average number of transportation problems solved.

Conditional Penalty	Capacity Improvement				
	None	Type I	Type II	Type III	Type IV
None	**932**	**362**	269	258	247
Type I	**313**	**200**	169	172	158
Type II	**276**	232	215	209	190
Type III	**189**	158	145	145	136
Type IV	145	122	113	111	110

the CPLEX MIP solver. For the MIP solver, the *FCTP* was formulated using a set of continuous variables $\{x_{ij}\}$ and a set of binary variables $\{y_{ij}\}$. The feasible region was defined using constraints (2) and a set of "forcing" constraints "$x_{ij} \leq u_{ij}y_{ij}\ \forall\ (i, j) \in A$," where u_{ij} is given by Eq. (5). Therefore, the CPLEX MIP solver provided an independent benchmark for the procedures discussed in this paper. The results of these computational tests are given next.

6.2. Results

For each of the 25 possible combinations of conditional penalties and capacity improvement, the number of transportation problems solved and the CPU time for the branch-and-bound algorithm was recorded for each of the 48 test problems. Tables 2 and 3 show, respectively, the average number of transportation problems solved and the average CPU time (in seconds) for the 48 test problems. The numbers in boldface in Tables 2 and 3 indicate results based on techniques previously reported in the literature. The Type I, II, and III conditional penalties without capacity improvement are described in [2, 5, 6, 8, 11, 12, 18, 20, 26]. The Type I capacity improvement technique without conditional penalties are discussed in [15–17, 22, 23, 25]. The combination of Type I conditional penalties with Type I capacity improvement was introduced in [14]. The nonboldface numbers in Tables 2 and 3 are new results based on the techniques presented in this paper.

In Table 2, the entry in the upper left-hand corner indicates that, on average, 932 (linear) transportation problems needed to be solved when neither conditional penalties nor capacity improvement was used. [In comparison, an average of 297 subproblems had to be solved when the CPLEX MIP solver was used.] The first column of Table 2 shows the effect of conditional penalties alone (i.e., without capacity improvement). This column shows that the conditional penalties contributed significantly in fathoming subproblems in the branch-and-bound procedure. Also, as expected, the Type IV conditional penalty was stronger than any of the other types of conditional penalties. The first row in Table 2 shows the effect of capacity improvement alone (i.e., without conditional penalties). The table shows that greater fathoming was achieved when

Table 3. Average CPU seconds.

Conditional Penalty	Capacity Improvement				
	None	Type I	Type II	Type III	Type IV
None	**5.61**	**4.32**	3.27	3.11	3.22
Type I	**4.40**	**3.13**	2.55	2.56	2.58
Type II	**3.51**	3.36	3.03	2.94	2.86
Type III	**2.67**	2.55	2.24	2.24	2.24
Type IV	2.28	2.11	1.89	1.83	1.89

capacity improvement was used; and the Type II, III, and IV capacity improvement (based on a concave relaxation) produced tighter coefficients than the Type I capacity improvement (based on a linear relaxation). The remainder of the table shows the effects when both conditional penalties and capacity improvement were used. This part of the table shows that it was more beneficial to use both techniques together than to use either one of the techniques alone. The greatest reduction was achieved using Type IV conditional penalties together with Type IV capacity improvement. In this case, an average of 110 transportation problems had to be solved for the 48 test problems. This represents an 88% reduction compared to the branch-and-bound procedure without conditional penalties or capacity improvement.

To quantify this improvement further, let η_{TP} denote the percent reduction in the average number of transportation problems solved using Type IV penalties and Type IV capacity improvement compared to no penalties or capacity improvement. That is, let

$N_{\mathrm{TP}}^{0,0}$ = average number of transportation problem solved without penalties or capacity improvement,

$N_{\mathrm{TP}}^{\mathrm{IV,IV}}$ = average number of transportation problem solved with Type IV penalties and Type IV capacity improvement.

Then, η_{TP} is defined as

$$\eta_{\mathrm{TP}} = \frac{(N_{\mathrm{TP}}^{0,0} - N_{\mathrm{TP}}^{\mathrm{IV,IV}})}{N_{\mathrm{TP}}^{0,0}} \times 100. \tag{35}$$

As mentioned in the previous paragraph, for the entire set of test problems, $\eta_{\mathrm{TP}} = 88\%$. This improvement increased as the test problems got larger. Specifically, for the smaller test problems (sets 1–7 in Table 1), $\eta_{\mathrm{TP}} = 79\%$; for the medium-sized problems (sets 8–12) $\eta_{\mathrm{TP}} = 92\%$; and for the larger problems (sets 13–16), $\eta_{\mathrm{TP}} = 95\%$.

Furthermore, the improvement in η_{TP} also increased as the test problems became harder. Here, "hardness" was measured by the ratio of the fixed charge objective function coefficient to the unit cost objective function coefficient averaged over the arcs in the network. Let this ratio be denoted by h. That is,

$$h = \frac{1}{|A|} \sum_{(i,j)\in A} \frac{f_{ij}}{c_{ij}}. \tag{36}$$

Problems with a larger value of h generally have a greater objective function value gap between problem P and the LP relaxation $\bar{P}$. Thus, problems with a larger value of h tend to be harder to solve.

To control for problem size, the largest test problems (i.e., sets 13–16) were selected. The average value of h for sets 15 and 16 was 0.45. Hence, sets 15 and 16 can be classified as the "easier" problems among the group of larger test problems. For these easier test problems, $N_{\mathrm{TP}}^{0,0}$ averaged 189 and $\eta_{\mathrm{TP}} = 78\%$. In contrast, the average value of h for sets 13 and 14 was 40.0. This means that, among the group of larger test problems, sets 13 and 14 can be classified as the "harder" problems. For the harder test problems, $N_{\mathrm{TP}}^{0,0}$ averaged 1516 and $\eta_{\mathrm{TP}} = 97\%$. Thus, the improvement in η_{TP} was greater for the harder problems.

Table 3 shows similar patterns for the average CPU time. An average of 5.61 s was required when neither conditional penalties nor capacity improvement was used. [This compares to an average CPU time of 3.14 s for the CPLEX MIP solver.] The first column of Table 3 shows the effect of conditional penalties alone; and the first row shows the effect of capacity improvement alone. Here, the most efficient conditional penalties were the Type IV penalties. For capacity improvement, the Type I capacity improvement was more efficient than not using any capacity improvement; and the Type II, III, and IV capacity improvement methods were more efficient than the Type I method. However, the average CPU times for the branch-and-bound algorithm using the Type II, III, and IV capacity improvement methods were roughly the same. Thus, although the Type IV improved capacity coefficients were tighter than the Type III coefficients, this benefit was offset somewhat by the fact that greater computational efforts were required to calculate the Type IV coefficients. This same relationship was true between the Type III and the Type II improved capacity coefficients.

Another pattern brought out by Table 3 is that, once again, it is more efficient to combine conditional penalties and capacity improvement. In this case, the lowest CPU time, on average, occurred when the Type IV conditional penalties were used with the Type II, III, or IV capacity improvement. Using these types of conditional penalties and capacity improvement yielded a 67% reduction in CPU time compared to the branch-and-bound algorithm without conditional penalties or capacity improvement.

Moreover, as with the number of transportation problems solved, the greatest improvement in CPU time occurred with the largest, hardest problems [where "hardness" was measured using the ratio h defined in Eq. (36)].

7. SUMMARY

This paper has examined branch-and-bound solution methods for the fixed charge transportation problem (*FCTP*). A distinguishing feature of our work was the use of a concave relaxation in place of the standard linear relaxation of subproblems in the branch-and-bound procedure. Parametric analysis of the concave relaxation was used to generate conditional (i.e., "up" and "down") penalties. A new conditional penalty, which is stronger than other conditional penalties given in the literature for the *FCTP*, was derived. In addition, the concave relaxation was used to generate coefficients in a capacity improvement (i.e., "range reduction" or "domain contraction") procedure. These coefficients are tighter than the capacity improvement coefficients derived from a linear relaxation of the subproblems. Computational experiments, based on a set of standard *FCTP* test problems, indicated that (i) the conditional penalties proposed in this paper are more efficient than previously proposed penalties, (ii) the capacity improvement techniques based on a concave relaxation are more efficient than those based on a linear relaxation, (iii) combining conditional penalties with capacity improvement is more efficient than using either technique alone, and (iv) the percent improvement in solution time achieved by using conditional penalties and capacity improvement is greater for the larger, harder test problems.

In closing, we note three areas in which the techniques described in this paper can be applied to problems that are more general than the *FCTP*. First, conditional penalties and capacity improvement can be applied to network flow problems with arbitrary concave arc cost functions (Bell and Lamar [3]) as well as to problems involving the minimization of a concave function over a polytope (Lamar [15]). Second, the Type I through Type IV penalties presented in this paper can also be combined with other types of conditional penalties, such as those proposed by Bretthauer [4]. The Bretthauer penalties give a lower bound to the objective function value increase when a Tuy [27] cut is applied to the LP relaxation of the problem. The Bretthauer penalty can be made even tighter by basing it on the types of concave relaxations described in

Section 3 (rather than on a linear relaxation). Third, a promising area of future research is to combine the techniques presented in this paper with other solution procedures. For example, conditional penalties and capacity improvement can be used in conjunction with valid inequality and cutting plane techniques (see, for example, Nemhauser and Wolsey [19], Savelsbergh [24], and Van Roy and Wolsey [28]). Here, the improved capacity coefficients may produce stronger valid inequalities which, in turn, may yield stronger conditional penalties and tighter improved capacity coefficients.

ACKNOWLEDGMENTS

The authors wish to thank the Associate Editor and the two anonymous referees for their comments. Their suggestions have resulted in an improved paper. The authors also appreciate Dr. Martin Savelsbergh's comments on an earlier version of this paper.

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