# Formulas for the motion of a sensing bar due to nearby masses [1] 1

Lang P. Withers Jr.
The MITRE Corporation, McLean, VA
Jan. 12, 2011

## 1 Introduction

As our frame of reference, we begin with Cartesian coordinates in 3-space $\mathbb{R}^3$. Assume the frame is non-accelerating (inertial). Suppose there is a mass $m$ at a fixed point in space $\mathbf{r}' = (x', y', z')$. This is a point source. It produces a gravity field around it, which can be sensed by a test mass at another point in space. How strong and in what direction is the tug of the field as it acts on the test mass? If two test masses sense the field, how different are the tugs they sense? We are asking for the values of

- the gravitational field $\mathbf{g} = (g_1, g_2, g_3)$. We can also write this as just $g_a$, understanding that the index $a$ runs over the three space coordinates: $a = 1, 2, 3$.
- its spatial gradients $\nabla\mathbf{g} = \frac{\partial g_a}{\partial r_b}$, for $a = 1, 2, 3$ and $b = 1, 2, 3$.

at any other point $\mathbf{r} = (r_1, r_2, r_3) \equiv (x, y, z)$. The distance between the source and sense points in 3-space is given by $|\mathbf{r}' - \mathbf{r}| = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$. To compute these field values for any distribution of point source masses, we merely add up (superpose) the fields from all the masses. We show how to do this in Section 2.

The gradient of the field can be interpreted as the *tidal acceleration tensor* [1, p.448]. It represents the differences in gravity tugs in the three basic directions, which two nearby particles feel pulling them apart or pushing them together. Our tides are caused by such differences in the Moon's (and Sun's) tugs on masses at opposite sides of the earth. For example, when the two particles are displaced from each other by a small distance along the $y$ direction ($b = 2$), the difference in gravity tugs along the $x$ direction ($a = 1$) is $\frac{\partial g_2}{\partial r_1}$. So this so-called tidal force can be measured by sensing how two test masses move, say inside a gradiometer.

What does this tidal tensor tell us? Suppose you are at one point in space $\mathbf{r}_0$, and you move by a relatively small increment

$\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$ to the point $\mathbf{r} = \mathbf{r}_0 + \Delta\mathbf{r}$. Then how much does the gravity field change? The tensor matrix gives us the answer: the change is about

$$\Delta\mathbf{g} = \nabla\mathbf{g} \cdot \Delta\mathbf{r}.$$

This is clearer in tensor notation:

$$\Delta g_a = \frac{\partial g_a}{\partial r_b}\Delta r_b. \tag{1}$$

(For uncluttered notation, we are using Einstein's sum convention: when a term has a repeated index ($b$ in this case), the sum is taken over all values of that index.) The signs of the tidal tensor components tell us even more.

Gravity only attracts (there is no anti-gravity particle as far as we know). So the force field at each point in space can only be a positive vector in the direction of attraction. Even the unknown particles of dark matter only attract. But the differences in attraction from point to point (tidal forces) can be positive or negative, of course. Formula (1) allows this as it should. (Distances in space are like this too, only positive. But their differences can be negative.)

Consider two small test masses here, both moving independently in the earth's gravity field. The sign of the tidal difference in the gravity force (1) tells us whether the masses are being pushed together, or pulled apart. For example, put the masses at $\pm\Delta z$ on the vertical $z$-axis ($r_3$-axis) pointing down in the positive direction. If the field gradient $\frac{\partial g_z}{\partial z}$ at their center is positive, then the lower mass experiences a tidal difference $\Delta g_z = +\frac{\partial g_z}{\partial z}\Delta z$, drawn as a vector pointing down in the $+z$ direction. The attraction of the external gravity field is stronger than at the center 0. The upper mass experiences a tidal difference $\Delta g_z = -\frac{\partial g_z}{\partial z}\Delta z$, drawn as a vector pointing up in the -$z$ direction (the gravity attraction is weaker). The two arrows point away from center. This is tension, pulling them apart (this may be how tensors got their name). But if the field gradient $\frac{\partial g_z}{\partial z}$ is negative, then the signs of the two tidal differences reverse. Then the downward gravity is stronger at the upper mass, and weaker at the lower mass. The two arrows point in toward the center. This is compression, bringing the two masses together.

Another way to use the two test masses is to fasten them to the ends of a rigid bar that turns on a pivot. Then the tidal force

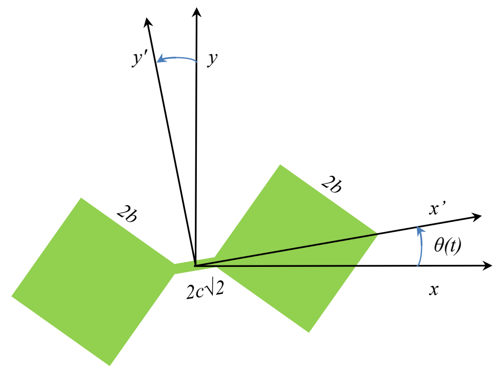


Figure 1: Gravity-sensing bar on a pivot. The bar is $2a$ thick in the $z$-dimension, normal to this page. This twin-diamond model idealizes the actual bar somewhat. The instrument contains two bars like this one in parallel planes. They have a common pivot axis normal to the planes, but are oriented at right angles to each other. An external mass rotates the bars slightly in opposite directions, producing a scissoring effect.

difference applies a torque to the bar, and it will rotate. See Figure 1.

Formula (1) is just the first-order Taylor expansion. More generally, to move from $\mathbf{r}_0$ to $\mathbf{r} = \mathbf{r}_0 + \Delta\mathbf{r}$, the expansion is

$$\boxed{g_a(\mathbf{r}) = g_a(\mathbf{r}_0) + \left.\frac{\partial g_a}{\partial r_b}\right|_{\mathbf{r}_0} \Delta r_b + \frac{1}{2!}\left.\frac{\partial^2 g_a}{\partial r_b \partial r_c}\right|_{\mathbf{r}_0} \Delta r_b \Delta r_c + \cdots} \quad (2)$$

In this note, we calculate the above Taylor coefficients (derivatives of the field) for a point source mass, based on Newton's theory of gravity. We will take $\mathbf{r}_0$ as the center of mass (CM) of a sensing bar on a pivot. We will see that the coefficients essentially only depend on the source mass and position, so that the sensor field at each point $\mathbf{r}$ of the bar depends only on the displacements $\Delta\mathbf{r}$ from the CM. The most important coefficients for us are those in the first-order term, the curvature or tidal acceleration components $\frac{\partial g_a}{\partial r_b}$. Moreover, we will see that the equation of motion of the pivoted bar can be set up using the expansion (2).

## 2 Tidal tensor components and beyond

We can begin with the Newtonian scalar potential $\phi$ at $\mathbf{r}$ due to the point mass at $\mathbf{r}'$: [2]

$$\phi(\mathbf{r}) = -\frac{Gm}{|\mathbf{r}' - \mathbf{r}|}, \tag{3}$$

where Newton's constant of universal gravitation is $G = 6.67428 \pm .00067 \times 10^{-11}$ in units of $\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$.

Then at any given point in space, $\mathbf{r}$, the gravity field (the tug on a unit test mass) has a strength in each of the three coordinate directions given by the gradient of this potential $\phi$ (its rate of change as we move slightly in each direction), a vector

$$\mathbf{g} = -\nabla\phi(\mathbf{r}) = \frac{Gm}{|\mathbf{r}' - \mathbf{r}|^3}(\mathbf{r}' - \mathbf{r}), \tag{4}$$

where the gradient operator is $\nabla = \nabla_{\mathbf{r}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. A simpler expression results if we put $\mathbf{s} = \mathbf{r}' - \mathbf{r}$, the distance vector between the two points, write $s = |\mathbf{s}|$ for the distance, and denote the unit vector (of direction cosines) pointing to the source at $\mathbf{r}'$ by $\hat{\mathbf{s}} = (\mathbf{r}' - \mathbf{r})/|\mathbf{r}' - \mathbf{r}| = \left(\frac{s_x}{s}, \frac{s_y}{s}, \frac{s_z}{s}\right)$:

$$\mathbf{g} = -\nabla\phi(\mathbf{r}) = \frac{Gm}{s^2}\hat{\mathbf{s}}. \tag{5}$$

This is Newton's familiar inverse-square law of universal gravitation. The field has a positive sign, indicating that the force exerted at the observation point $\mathbf{r}$ attracts us to the source at $\mathbf{r}'$.

To make our calculations easier, from now on we use tensor notation, plain and simple. Using Einstein's sum convention (any term that has products with the same index is to be summed over that index's values), we can rewrite Newton's inverse-square law (5) as

$$-\phi_a \equiv -\nabla_a\phi(\mathbf{r}) = g_a = \frac{Gm}{s^2}\left(\frac{s_a}{s}\right), \tag{6}$$

This is the center-of-mass or dipole coefficient in the Taylor expansion (2) of $\phi$.

We know that any well-defined physical vector quantity has the same meaning or role, regardless of the coordinate frame we happen

[2] Introduced after Newton by Clairau(l)t, Laplace, and Lagrange. My thanks go to Kieran Carroll for this point.

to see it in at any instant in time. (If we watch a vector $\mathbf{v}(t)$ at two instants of time, and consider its rate of change over time as a new vector $\dot{\mathbf{v}}$, then strange dynamic effects may occur in changing frames, as we'll see later. But here we freeze everything at one instant in time and consider different frames of reference.) The position vector $r_a$ of a particle in a Cartesian frame is a familiar example. If we apply a rotation $R_{ab}$ (or translation) to bring the frame to line up with another frame, the vector simply transforms to

$$r'_a = R_{ab} r_b, \tag{7}$$

and it preserves its identity as the position vector. (Its length $r = \sqrt{r_a r_a}$ is a true scalar, being invariant under change of frames.) The rotation $R_{ab}$ could be given, say, as a product of successive rotations by three independent Euler angles (about the $z$ axis, then the new $y$ axis, then the new $z$ axes).

An important detail to note, if we are ever to get all our $\pm$ signs correct, is that equation (7) always represents a passive transformation. We view a fixed point (and read out its coordinates) in the initial and rotated frames. An active transformation keeps the same frame and rotates the point. (Suppose you have a wheel with a crayon mark, and you view it in a camera. You get the same picture if you actively rotate the wheel by some angle, or passively rotate the camera frame the opposite way by the same angle.) For example, consider a rotation by the angle $\theta$ about the $\hat{\mathbf{z}}$ axis. (By convention, a positive angle means a c.c.w., i.e. anti-clockwise, rotation.) The passive rotation operator to use in equation (7) is given by

$$R_3(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

But the active rotation operator (to turn a point vector within a fixed frame) is $R_3(-\theta) = R_3^t(\theta)$, its transpose.

(In general, tensors transform in two ways: contravariantly and covariantly. The two typical examples are a differential (contra-) and a derivative (co-), such as a gradient. But the two ways are identical for rotations in Cartesian coordinates, so we do not need this distinction in this paper.)

Now we take the partial derivative $\partial/\partial_a$ of each component $g_b$ in formula (4), and calculate

$$-\phi_{ab} = -\nabla_a \nabla_b \phi(\mathbf{r}) = \nabla_b \mathbf{g}_a = -\frac{Gm}{s^3}\left(\delta_{ab} - 3\frac{s_a s_b}{s^2}\right), \tag{9}$$

where $\delta_{ab}$ is 1 when $a = b$ and 0 otherwise. This tensor is essentially an inverse-cube law for the tidal acceleration. It is the quadrupole coefficient in the Taylor expansion (2) of $\phi$.

For example, if the two locations (where the point mass is and where we are going to sense the field) are both on the $z$-axis, we easily find that the cross-terms are all zero, but

$$\phi_{xx} = \phi_{yy} = -\frac{Gm}{s^3} \tag{10}$$

and

$$\phi_{zz} = \frac{2Gm}{s^3}. \tag{11}$$

This shows that the differential gravity tugs are proportional to the inverse-cube of the distance away. This is true in general, since we can always rotate our axes so that both points are lined up parallel to the $z$-axis.

**Second-order tensors.** Here is how a second-order tensor, to be a tensor, must transform under any rotation $R_{ab}$ of the frame of reference.

$$\phi'_{ab} = R_{ac} R_{bd} \phi_{cd}. \tag{12}$$

Note that $R_{ac}$ is the same $3 \times 3$ rotation as $R_{bd}$, just with different index labels!

As an example of how tensors transform, consider equation (1) again:

$$\Delta g_a = \frac{\partial g_a}{\partial r_b} \Delta r_b.$$

Let us transform the left and right hand sides of this equation after a passive rotation $R_{ab}$, taking baby steps backwards:

$$\begin{aligned}
\Delta g'_a &= \left(\frac{\partial g_a}{\partial r_b}\right)' \Delta r'_b \\
R_{ac} \Delta g_c &= R_{ac} R_{bd} \frac{\partial g_c}{\partial r_d} R_{be} \Delta r_e \\
R_{ac} \Delta g_c &= R_{ac} R_{bd} R_{be} \frac{\partial g_c}{\partial r_d} \Delta r_e \\
R_{ac} \Delta g_c &= R_{ac} \delta_{de} \frac{\partial g_c}{\partial r_d} \Delta r_e
\end{aligned}$$

$$
\begin{aligned}
R_{ac}\Delta g_c &= R_{ac}\left(\frac{\partial g_c}{\partial r_e}\Delta r_e\right) \\
\Delta g_c &= \frac{\partial g_c}{\partial r_e}\Delta r_e,
\end{aligned}
$$

which brings us back to the equation in the original frame with no harm done. Our algebra is consistent. This simple algebra proves important below, when we have to compute mixed quantities from two frames.

A word about matrices: tensors are simpler! In general, the matrix product of two rank-2 tensors $A$ and $B$ is $A_{ab}B_{bc}$. The reader knows the inverse of a rotation matrix is given by its transpose. This is because any rotation $R$ preserves lengths (it is an isometry), so its columns (and rows) are orthonormal. To transpose a tensor, we just exchange its indices, in whatever context we find it: $R^t_{ab} = R_{ba}$. For example, we have $R^t_{ab}R_{bc} = R_{ba}R_{bc} = \delta_{ac}$, the identity tensor. The last equality simply states that the columns of $R$ are orthonormal. One could forget about matrix products, and just begin with the last equality to express the orthonormality of $R$!

Let us compare equation (12) to a change of basis for the gradient matrix $\phi_{cd}$, given by the similarity transformation $\phi' = S^{-1}\phi S$ in linear algebra. What is $S$ in this case? It is the *active* rotation that expresses the new basis in terms of the original basis, given by $S = R^t$ (as noted above). That is, $\phi'_{ab} = R_{ac}\phi_{cd}R^t_{db}$. One may easily verify that this is equivalent to (12). But the correct linear-algebraic expression to transform the basis of a vector is not $r' = Sr$. It applies the *passive* rotation $r' = Rr$, just as in (7). In contrast, the tensor transforms simply apply passive rotations, for a tensor of any rank. The tensor version does not make us think so hard.

**Third and fourth order gravity tensor components.** For close range cases, where the torque baseline of the sensing bar is not much smaller than the distance from the source masses to the bar, we may need the next two terms in the Taylor series (2). From (9), the next derivative is readily found:

$$
\begin{aligned}
\nabla_b\nabla_c\mathbf{g}_a &= -\phi_{abc} = -\nabla_a\nabla_b\nabla_c\phi(\mathbf{r}) \\
&= \frac{3Gm}{s^4}\left(\delta_{bc}\frac{s_a}{s} + \delta_{ca}\frac{s_b}{s} + \delta_{ab}\frac{s_c}{s} - 5\frac{s_a s_b s_c}{s^3}\right). \quad (13)
\end{aligned}
$$

*Comments:* This term clearly falls off as the inverse-fourth power of the sensor-source separation distance $s$. It is the octupole coefficient in the Taylor expansion (2) of $\phi$. The direction cosines $s_a/s$ of the distance vector $\mathbf{s}$, for $a = 1, 2, 3$ in this reference frame are dimensionless numbers between $\pm 1$. Note that the formula keeps the same value when the three indices $a, b, c$ are exchanged (permuted) in any way, as a third partial derivative should. This is a third-order tensor with $3^3 = 27$ components, one for each choice of $a, b, c \in \{1, 2, 3\}$.

By now, you can guess how a third-order tensor should transform under any rotation $R_{ab}$ of the frame of reference:

$$\phi'_{abc} = R_{ad} R_{be} R_{cf} \phi_{def}. \tag{14}$$

This is the defining property of a true tensor, that it preserves its form in this fairly simple way when we rotate the frame of reference. Again, you see that the "handles" (indices) of the rotation are relabeled each time it is applied, so that the rotation operates across another dimension of the tensor. Again the Einstein sum shorthand makes it easy to see which handle is being summed over in each instance: look for the same letter twice.

In relativity, a tensor transforms under a Lorentz transformation $\Lambda_{\alpha\beta}$ (to change the spacetime frame of reference) in the same manner as for rotations. Besides our three space dimensions, e.g. $a = 1, 2, 3$, the dimension of time is now included. Each index runs over the four dimensions, e.g. $\alpha = 0, 1, 2, 3$. (There is a new distance metric to measure spacetime intervals, etc.) But in any event, we are borrowing this tensor notation to make life easy and straightforward. There's a reason for this alphabet-soup notation Einstein adopted!

The reader can now compute the next Taylor coefficient of the gravity field about the bar's center of mass, $\nabla_b \nabla_c \nabla_d \mathbf{g}_a = -\phi_{abcd}$. It is a fourth-order tensor with $3^4 = 81$ elements that fall off as the inverse fifth power of the distance $s$ from source to sensor. From (13), we compute

$$\begin{aligned}
\nabla_b \nabla_c \nabla_d \mathbf{g}_a &= -\phi_{abcd} \equiv -\nabla_a \nabla_b \nabla_c \nabla_d \phi(\mathbf{r}) \\
&= -\frac{3Gm}{s^5} \Big\{ -\left(\delta_{bc}\delta_{ad} + \delta_{ca}\delta_{bd} + \delta_{ab}\delta_{cd}\right) \\
&\qquad +5 \left( \delta_{bc} \frac{s_a}{s}\frac{s_d}{s} + \delta_{ca}\frac{s_b}{s}\frac{s_d}{s} + \delta_{ab}\frac{s_c}{s}\frac{s_d}{s} \right. \\
&\qquad \left. + \delta_{ad}\frac{s_b}{s}\frac{s_c}{s} + \delta_{bd}\frac{s_a}{s}\frac{s_c}{s} + \delta_{cd}\frac{s_a}{s}\frac{s_b}{s} \right) \\
&\qquad -5 \cdot 7 \frac{s_a s_b s_c s_d}{s^4} \Big\}.
\end{aligned}$$

(15)

This fourth derivative is the $2^4$ multipole coefficient in the Taylor expansion (2) of $\phi$. A pattern seems to be emerging, but we will stop here. Again we can check that each part of the formula is invariant under permutations of the indices $a, b, c, d$.

*Comments on convergence:* The important implication for the convergence of the Taylor series (2) for the gravity field in our short-range sensing application is this: that higher order terms matter as the bar length approaches the standoff distance to the sources. A rough estimate may be helpful. The coefficient (derivative) of the $n$th term depends inversely on the $(n+1)$-power of the average distance $R$ from source masses to the bar's center-of-mass (CM). But the product of the variables in the $n$th term is essentially the $n$th power of the distance $r$ from the CM to the bar's elements. So the $n$th term is roughly proportional to $\frac{1}{R} \cdot \left(\frac{r}{R}\right)^n$. So if the two types of distances are comparable (e.g. the sources are $R = 20$cm away, the bar is $2r = 10$cm end-to-end), then the higher-order terms in the Taylor series matter.

When we scan the source masses with the bar and want to recover those masses from the sensed torques on the bar, the accuracy matters. This is basically because the gravity signal (the torque measured from the bar) falls off so rapidly, as the square of the inverse distance. So the dynamic range of the superposed signals that must be separated is large. We want to model the strong contributions from near sources, say, at $R = 20$cm more accurately, so that the modeling error does not mask the weaker contributions from, say, $R = 2$m away, which may only be about the level of the error in the stronger. (To subtract the background masses, the empty container, without the source masses of interest, is also scanned initially.)

## 3 Euler's equations of motion for a rotating bar

In this section, we review the classical mechanics for our rotating bar, taking a careful look at the space and body frames of reference. We derive our basic equation of motion (22) below in the body frame. But in our special case, due to the pivot constraints, we see that it is the same equation of motion in the space frame. This fact will keep our calculations in the next section simple, since our gravity potential is given in the space frame.

A torsion-bar gradiometer senses and measures the torque on a small bar which is free to rotate by small angles about a pivot. Suppose the sensing bar itself has mass density $\sigma(\mathbf{r})$. This density may be uniform (constant) inside the surface that contains the bar. Then a volume element $dV$ inside the bar has mass

$$dm = \sigma(\mathbf{r})dV. \tag{16}$$

The bar is a rigid solid, so the internal gravitational forces of the bar all cancel, and have no effect on the motion of the bar. The bar moves in response to forces from outside sources or external forces, which in our case come from the net gravity field of the masses outside the bar. Assume the bar is constrained to pivot about a fixed point, namely its center of mass or first moment $\mathbf{r}_0 = \int \mathbf{r}\ \sigma(\mathbf{r})dV$. The origin of our space frame and our body frame are both taken at the center of mass.

Let us consider how the bar moves. When a force vector $\mathbf{F}$ acts upon a particle of mass $m$ in an inertial frame fixed in space, Newton's equation of motion is $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, or $F_a = m\dot{v}_a = \frac{dp_a}{dt}$, for space components $a = 1, 2, 3$ of the linear momentum $\mathbf{p}$. The rotary force or torque $\boldsymbol{\tau}$ on the particle with respect to a reference point is $\tau = \mathbf{r} \times \mathbf{F}$. For the "space" (inertial, fixed) frame, the analog of Newton's equation is $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$, for the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ about that point. For a rigid body, it is well known that $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$; that is, $L_a = I_{ab}\omega_b$, where $\mathbf{I} = I_{ab}$ is the (second) moment of inertia tensor about the point, defined by [2]:

$$I_{ab} = \int \left(r^2\delta_{ab} - r_a r_b\right)\ \sigma(\mathbf{r})dV. \tag{17}$$

In general, since the position coordinates of the particles that make up the body are changing with time, the components $I_{ab}$ of the moment of inertia are changing too.

But in the body (rotating) frame, a coordinate frame fixed in the moving rigid bar, the body's moment of inertia tensor components remain constant. Euler's equations of motion will allow us to change to the body frame (see [2], chapter 4). First, we must distinguish between how the angular momentum vector $\mathbf{L}$ changes from moment to moment, as seen in the space frame and as seen in the rotating body frame:

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{L} \tag{18}$$

Note that in this formula, the vectors on both sides (regardless of the frame in which they are originally *defined*) must be given in coordinates with respect to a *single* frame of reference, such as the rotating body frame. (Basically, this formula says that $\dot{\mathbf{L}}$ does not behave as a true vector. For if it did, the formula would just be an equality, one and the same vector as viewed in any frame. But an extra term $\boldsymbol{\omega} \times \mathbf{L}$ is needed to correct for the frame-of-origin change. Also, this formula is quite general, and is true for the rate of change of any vector over time as seen in the two frames, not just that of $\mathbf{L}$. In relativity, an extra term must similarly be added to keep the derivative changing covariantly under Lorentz boosts.) Since for the space frame, $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$ (Newton's law of motion applies), and since in the body frame $\dot{L}_a = I_{ab}\dot{\omega}_b$ (again, $I_{ab}$ does not change in the body frame), equation (18) becomes

$$\boldsymbol{\tau} = \boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} \tag{19}$$

or in tensor notation[3]

$$\tau_a = I_{ab}\dot{\omega}_b + \epsilon_{abc}\,\omega_b I_{cd}\omega_d. \tag{20}$$

For the body frame defined by the bar's own principal axes, the cross-moments of inertia are 0 (put $I_{ab} = \delta_{ab} I_a$), and Euler's equations of motion simplify as

$$\begin{aligned} \tau_1^{(1)} &= I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 \\ \tau_2^{(1)} &= I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 \\ \tau_3^{(1)} &= I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2. \end{aligned} \tag{21}$$

Finally, for our bar constrained to pivot about the $z$-axis, $\omega_1 = \omega_2 = 0$, so Euler's equations reduce to one equation of motion,

$$\boxed{\tau_3 = I_{33}\dot{\omega}_3 = I_{33}\ddot{\theta},} \tag{22}$$

[3]In tensor notation, any vector cross product $\mathbf{x} = \mathbf{y} \times \mathbf{z}$ can be written conveniently as

$$x_a = \epsilon_{abc} y_b z_c,$$

where $\epsilon_{abc}$ is a simple tensor (of order 3) that takes values 0 or $\pm 1$. It is defined by $\epsilon_{abc} = 0$ if any two indices are the same, and otherwise it is the sign of the permutation of the three indices in their given order. For example, for $a = 2$, $\epsilon_{233} = 0$, $\epsilon_{231} = +1$, $\epsilon_{213} = -1$. This useful little device is known as the Levi-Civita tensor.

where $\theta = \theta(t)$ is the angle of rotation (between the $x$-axis of the body with respect to the $x$-axis of the space frame) at time $t$.[4]

This equation has a physical property worth noting. Both sides of this equation of motion are actually invariant under the (imagined, instantaneous) rotations about the $z$-axis that take us from space to body frame at any given time. The pivot constraints imply that the axis of rotation, which we have taken as the $z$-axis, is fixed and common to both the space and body frames. So the moment of inertia about the $z$-axis, $I_{33} = \int (r_1^2 + r_2^2)\ \sigma(\mathbf{r})dV$, is constant in time, even in the space frame (unlike $I_{11}$ and $I_{22}$ - and the cross term $I_{12}$ becomes non-zero in the space frame). Recall that the torque vector $\tau_3\hat{\mathbf{z}}$ was defined in the space frame. It is clearly invariant under the frame rotation about $\hat{\mathbf{z}}$, so $\tau_3$ has the same value in either frame at any given time. Also $\dot{\omega}_3$ is the same in both frames, since $\dot{\boldsymbol{\omega}} = \dot{\omega}_3\hat{\mathbf{z}}$ is a true vector and is invariant here under the rotation about $\hat{\mathbf{z}}$. So each quantity on either side of the equation (22) has the same value in either frame, at any given time. Therefore, we are free to use (22) in the space frame.

To use Newton's law of motion, $F = m\ddot{x}$, we must say what the force $F$ is, based on the given physical situation. Similarly, to use (22), we must specify the total torque $\tau_3$. This will be easy to do in the space frame. To this task we turn next.

## 4 Torque on a bar in the gravity field

We must compute the total torque on the bar, moving about its pivot at the center of mass (CM). What are the torques? The flexure spring exerts a counter-torque $-k\theta$ about the $z$-axis proportional to the twist angle $\theta$, where $k$ is the spring constant. (Here we do not include any frictional torque to dampen the bar oscillation, and broaden its resonance linewidth.) The gravitational torque is the sum of the torques due to the combined tug of the gravity field (from all outside source masses), acting on each mass element of the bar. The total torque vector $\tau$, then, as seen in the space frame, amounts to the spring torque plus the gravity torque integrated over

[4]Note that $\dot{\boldsymbol{\omega}}$ is a true vector. That is, $\dot{\boldsymbol{\omega}}_{\text{space}} = \dot{\boldsymbol{\omega}}_{\text{rotating}}$. This follows since (18) is true for any vector, so we can substitute $\boldsymbol{\omega}$ for $\mathbf{L}$, and use $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$.

the bar's volume:

$$\boldsymbol{\tau} = -k\theta\hat{\mathbf{z}} + \int \mathbf{r} \times \mathbf{g}(\mathbf{r})\ \sigma(\mathbf{r})dV.$$

So the total torque can be written as

$$\tau_a \quad = \quad -\delta_{3a}k\theta + \int \epsilon_{abc} r_b g_c(\mathbf{r})\ \sigma(\mathbf{r})dV. \tag{23}$$

We want to substitute the Taylor expansion (2) of $g_c(\mathbf{r})$ into the torque formula (23). The gravity gradients $\phi_{cd}$ at the bar's CM $\mathbf{r}_0$, the origin, are known to us in the space frame. But the coordinates of the points in the rigid bar are rotating when observed in the space frame. To get to the body frame coordinates, we must rotate the quantities in the torque expression (23), including the gravity field given in the space frame. In the body frame with origin at the bar's CM, the Taylor expansion (2) now becomes

$$\boxed{R^\theta_{ab} g_b(\mathbf{r}) = R^\theta_{ab} g_b(0) + R^\theta_{ac} R^\theta_{bd} \left.\frac{\partial g_c}{\partial r_d}\right|_0 r'_b + \frac{1}{2!} R^\theta_{ad} R^\theta_{be} R^\theta_{cf} \left.\frac{\partial^2 g_d}{\partial r_e \partial r_f}\right|_0 r'_b r'_c + \cdots} \tag{24}$$

Here body coordinates are primed (not to be mistaken for source terms!). Equivalently, we may easily write this expansion in terms of the partial derivatives of the negative potential $-\phi$ that we worked out in the space frame in Section 2:

$$\begin{aligned} g'_a(\mathbf{r}') &= R^\theta_{ab} g_b(\mathbf{r}) \\ &= -\left( R^\theta_{ab}\phi_a(0) + R^\theta_{ac} R^\theta_{bd} \phi_{cd}\big|_0 r'_b + \frac{1}{2!} R^\theta_{ad} R^\theta_{be} R^\theta_{cf} \phi_{def}\Big|_0 r'_b r'_c + \cdots \right) \\ &= -\left( \phi'_a(0) + \phi'_{ab}\big|_0 r'_b + \frac{1}{2!}\phi'_{abc}\Big|_0 r'_b r'_c + \cdots \right) \end{aligned} \tag{25}$$

Again, keep in mind that we must compute partials of $\phi'$ by rotating those of $\phi$, which we know. The bar element positions $r'_a$ are, of course, most naturally given in the bar's body frame.

In our case, the angle $\theta$ about the $\hat{\mathbf{z}}$ axis is always small. The rotation operator is approximated to first order in $\theta$ by

$$R^\theta_3 \quad \equiv \quad \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

or in the compact and pliable tensor style,

$$R^{\theta}_{ab} \;\approx\; \delta_{ab} + \epsilon_{3ab}\theta. \tag{27}$$

**First-order torque.** Let us substitute the first four terms of the gravity field expansion (24) or (25) into the torque formula (23) in succession, and also transform to the body frame. The total torque about the center of mass will be approximately the sum of these four terms. The constant, 0[th] order term produces no torque:

$$\begin{aligned}
\tau'^{(0)}_a &= \int -\epsilon_{abc} r'_b \phi'_c(0)\ \sigma(\mathbf{r}')dV' \\
&= \int -\epsilon_{abc} r'_b R^{\theta}_{cd}\phi_d(0)\ \sigma(\mathbf{r}')dV' \\
&= -\epsilon_{abc} R^{\theta}_{cd}\phi_d(0) \cdot \left( \int r'_b\ \sigma(\mathbf{r}')dV' \right) \\
&= -\epsilon_{abc} R^{\theta}_{cd}\phi_d(0) \cdot 0_b = 0_a,
\end{aligned} \tag{28}$$

since the CM is taken as the origin $\mathbf{0} = (0,0,0) = 0_b$. Because of the zeroes, we did not have to worry about the rotation of $\phi_d$ here.

Most of the torque comes from the gravity gradient in the first-order term of (25):

$$\begin{aligned}
\tau'^{(1)}_a &= \int -\epsilon_{abc} r'_b \phi'_{cd} r'_d\ \sigma(\mathbf{r}')dV \\
&= -\epsilon_{abc}\phi'_{cd} \int r'_b r'_d\ \sigma(\mathbf{r}')dV \\
&\approx -\epsilon_{abc}\left(\delta_{ce} + \epsilon_{3ce}\theta\right)\left(\delta_{df} + \epsilon_{3df}\theta\right)\phi_{ef} \int r'_b r'_d\ \sigma(\mathbf{r}')dV
\end{aligned} \tag{29}$$

Setting the term of order $\theta^2 \approx 0$, we obtain a tractable formula:

$$\boxed{\tau'^{(1)}_a = -\epsilon_{abc}\left(\phi_{cd} + \left[\epsilon_{3ce}\phi_{ed} + \epsilon_{3df}\phi_{cf}\right]\theta\right) \cdot \left( \int r'_b r'_d\ \sigma(\mathbf{r}')dV' \right).} \tag{30}$$

A significant aspect of this first-order gravity torque term is that the gravity field potential factor (at the CM) separates from the body's moments of inertia, which are constants in its own frame. This is why we changed from the space to the body frame.

Let us unpack this formula to see what is in it. Suppose the body axes are chosen as the principal axes of the bar. Then the second cross-moments of inertia are 0, and we can insert a $\delta_{bd}$ in (29) to select just the on-diagonal moments:

$$\begin{aligned}\tau_a'^{(1)} &= -\epsilon_{abc}\phi'_{cd}\delta_{bd}\int r'_b r'_d\ \sigma(\mathbf{r}')dV' \\ &= -\epsilon_{abc}\phi'_{bc}\int r_b'^2\ \sigma(\mathbf{r}')dV'. \end{aligned} \quad (31)$$

For example, the torque about the $z$-axis of the body is

$$\begin{aligned}\tau_3'^{(1)} &= -\phi'_{12}\int r_1'^2\ \sigma(\mathbf{r}')dV' + \phi'_{21}\int r_2'^2\ \sigma(\mathbf{r}')dV' \\ &= \phi'_{12}\int \left(r_2'^2 - r_1'^2\right)\ \sigma(\mathbf{r}')dV'. \end{aligned} \quad (32)$$

By (17), we have the second moment differences $I_{bb} - I_{cc} = \int(r_c'^2 - r_b'^2)\ \sigma(\mathbf{r}')dV'$. So we can write out all three components:

$$\begin{aligned}\tau_1'^{(1)} &= (I_2 - I_3)\,\phi'_{23} \\ \tau_2'^{(1)} &= (I_3 - I_1)\,\phi'_{31} \\ \tau_3'^{(1)} &= (I_1 - I_2)\,\phi'_{12}. \end{aligned} \quad (33)$$

(These are to be wed to the principal-axes Euler equations (21), also given in the body frame.) Note the minus sign beside the potential $\phi$ has been absorbed here. We see that to first order, each torque component $\tau_a$ is directly proportional to a gravity gradient component in the body frame. For our bar constrained to move only about the $z$-axis, only the torque about this axis matters. In terms of the gravity field which we know in the non-rotated, inertial space frame, evaluating (30) gives us this torque (to first order)

$$\boxed{\tau_3^{(1)} = (I_1 - I_2)\left[\phi_{12} + (\phi_{22} - \phi_{11})\theta\right].} \quad (34)$$

Suppose the space frame is not chosen with its $x$-axis to coincide with that of the bar at rest in no gravity field. For example, we will have two bars with perpendicular rest axes. Then $\theta$ (the angle between space and body $x$-axes) is large for one of the bars or armatures. We can put $\theta'(t) = \alpha + \theta(t)$ to allow any fixed angle $\alpha$ to the rest $x$ axis, and a small time-varying angle to the body $x$ axis. Then we can apply the exact rotation matrices to change coordinate frames, not their small angle approximation. If we compute

$\phi'_{12} = R_{1a}R_{2b}\phi_{ab}$ directly, from (32) we obtain the exact formula

$$\begin{aligned} \tau_3^{(1)} &= (I_1 - I_2)\left[\phi_{12}\cos 2\theta' + \frac{1}{2}(\phi_{22} - \phi_{11})\sin 2\theta'\right] \qquad (35) \\ &\approx (I_1 - I_2)\left[\phi_{12}\cos 2\alpha + \frac{1}{2}(\phi_{22} - \phi_{11})\sin 2\alpha\right] \\ &\quad + (I_1 - I_2)\left[-2\phi_{12}\sin 2\alpha + (\phi_{22} - \phi_{11})\cos 2\alpha\right]\theta. \qquad (36) \end{aligned}$$

and its small angle approximation. For the bar aligned at $\alpha = 0$, this reduces to (34). For the other bar mounted perpendicular to the first, with $\alpha = \pi/2$, this gives the same approximate torque (34), but with its sign reversed since $\cos 2\alpha = -1$.

**Equation of motion.** What effect does the torque have on the bar? Using formula (34) together with (22) and (23), we have a complete, first-order equation of motion for the bar's rotation angle $\theta(t)$:

$$I_3\ddot{\theta} = -k\theta + (I_1 - I_2)\left[\phi_{12} + (\phi_{22} - \phi_{11})\theta\right]. \qquad (37)$$

Since the external gravity field is static, we rewrite this as

$$\boxed{I_3\ddot{\theta} = -\kappa(\theta - \theta_0).} \qquad (38)$$

for the effective spring constant $\kappa = k - (I_1 - I_2)(\phi_{11} - \phi_{22})$, and the constant angle $\theta_0 = (I_1 - I_2)\,\phi_{12}/\kappa$. This has the general solution

$$\theta(t) = \theta_0 + A\sin(\omega_0 t + \varphi), \qquad (39)$$

for $\omega_0 = \sqrt{\kappa/I_3}$, and amplitude $A$ and initial phase $\varphi$ determined by initial conditions. This solution $\theta(t)$ is basically the signal we measure and sample, with noise added. Below we will see that $\theta_0$ is what we want to estimate or extract from this signal.

**Measuring the gradients.** In a nutshell, here is how the instrument works. Near each end of the niobium (Nb) bar, there is a coil of insulated niobium wire with a persistent circulating current [4]. By itself, each coil produces a stable magnetic field. But the field lines are excluded from the superconducting Nb bar at the low cryostat temperature about 5°K (Meissner effect). Niobium becomes superconducting below 9.27°K. The bar, moved by gravity, displaces the magnetic flux lines that go through the coil, inducing a back-emf in the coil. The current change is proportional to the angle $\theta(t)$ of the bar. A sensitive SQUID loop nearby measures the corresponding change in magnetic flux of the coil. (We have simplified and left out an intermediate transformer stage. See reference [4] for the actual circuit. Also, the SQUID measures the magnetic flux from both coils.) Finally, the instrument actually measures the *difference* of the magnetic flux in the upper and lower bars' loops. This is of course proportional to the difference of the upper and lower bar angles $\overline{\theta} - \underline{\theta}$ at each sampling time $t$. Since they are essentially equal but of opposite sign, this difference doubles the scaled angle.

The bar angle $\theta(t)$ given by the solution (39) has a simple power spectrum with two parts, a d.c. (constant) term $\theta_0$, and a resonance at frequency $\omega_0$ (about 7Hz = $14\pi$ rad/s). (No damping term is included, which limits and broadens the Lorentzian resonance line.) Consider the power spectrum of the measured angle difference $\left(\overline{\theta} - \underline{\theta}\right)(t)$. The amplitude of the d.c. (0-frequency) term measures the time-averaged difference $\left(\overline{\theta}_0 - \underline{\theta}_0\right)$. There will also be a closely spaced pair of resonance lines at the frequencies $\overline{\omega}_0$ and $\underline{\omega}_0$, which is not resolved by the spectrum estimate. So we measure their blurred average position $\hat{\omega}_0 = \frac{1}{2}\left(\overline{\omega}_0 + \underline{\omega}_0\right)$ from the spectrum.

Here we are considering the lower bar to be identical to the upper bar and mounted below it at a lower $z$ height. But it is turned in its $xy$-plane perpendicular to the upper, with space frame angle $\alpha = \pi/2$. Recall that in formula (36), the signs of $\phi_{12}$ reverse for the lower bar since $\cos 2\alpha = -1$. This means that the effective spring constants $\kappa$ are different for the upper and lower bars, due to the gravity gradients. Put $\Delta k = (I_1 - I_2)(\phi_{11} - \phi_{22})$. Using our relationships from (39), we have effective spring constants

$$\begin{aligned} \overline{\kappa} &= k - \Delta k \\ \underline{\kappa} &= k + \Delta k. \end{aligned}$$

Then let us put $C = (I_1 - I_2)\phi_{12}$. The measured d.c. amplitude in the power spectrum is

$$\begin{aligned} \left(\overline{\theta}_0 + \underline{\theta}_0\right) &= \frac{C}{\overline{\kappa}} + \frac{C}{\underline{\kappa}} = C\left(\frac{1}{k - \Delta k} - \frac{1}{k + \Delta k}\right) \\ &\approx \frac{2Ck}{k^2} = \frac{2C}{k} \equiv \frac{2(I_1 - I_2)\phi_{12}}{k}, \end{aligned} \tag{40}$$

where we assume $\Delta k \ll k$. To first order in $\Delta k$, the average angle difference removes the spring constant shift $\Delta k$. Our second possible measurement, the average resonant frequency, becomes

$$\begin{aligned} \tfrac{1}{2}\left(\overline{\omega}_0 + \underline{\omega}_0\right) &= \tfrac{1}{2}\left(\sqrt{\frac{\overline{\kappa}}{I_3}} + \sqrt{\frac{\underline{\kappa}}{I_3}}\right) \\ &= \tfrac{1}{2}\sqrt{\frac{k}{I_3}} \cdot \left(\sqrt{1-\Delta k/k} + \sqrt{1+\Delta k/k}\right) \\ &\approx \sqrt{\frac{k}{I_3}}. \end{aligned} \quad (41)$$

where we used the approximation $\sqrt{k \pm \Delta k} \approx \sqrt{k}\left(1 \pm \frac{\Delta k}{2k}\right)$. This measurement is useful for determining the actual spring constant $k$ at the pivots. But, up to first order in $\Delta k$, it contains no information about the gravity gradients.

A standard procedure is to apply a lowpass filter that rolls off at e.g. 3Hz. This excludes seismic noise, and the resonance at $\omega_0$. Intrinsic noise due to Brownian motion of the bar in a vacuum at low temperature, and SQUID amplifier noise are also present, as well as other error sources (temperature drift, misalignments, cross-axis coupling through the flexure pivot, etc.) [4][5]. To help remove $1/f^\nu$ noise at the lowest frequencies, a highpass filter to remove e.g. sub-.01Hz noise can also be applied. After filtering, the gravity signal we measure is the time-average of (40) to first order, or (49) to third-order.

**Example.** The various gradient tensor components are plotted for a two-dimensional mass layout upon a horizontal $8' \times 10'$ table-top in Figure 2. They are plotted for two sensor standoff distances, 25cm and 65cm. The masses are uniformly 10cm thick in the vertical direction, with densities as indicated. Using a nominal value of $\omega_0 = 7\text{Hz} = 14\pi$ rad/s, and $I_3$ from the twin-diamond rotor example at the end of Appendix B, we have $k = I_3\omega_0^2 = 276052740.569$ g cm$^2$/sec$^2$ (dyne-cm) = 27.605N-m; all per radian. For this example, suppose the field gradients are $\phi_{11} - \phi_{22} = 100\text{E} = 100 \times 10^{-9}/\text{sec}^2$, and $\phi_{12} = 25\text{E} = 25 \times 10^{-9}/\text{sec}^2$. Then we have the following exact values. The effective spring constants of the upper and lower bars are $\kappa = k \mp \Delta k = k \mp (I_1 - I_2)(\phi_{11} - \phi_{22})$, where $\Delta k = -0.0109$ dyne-cm/rad. That is, $\Delta k/k = -4 \times 10^{-11}$, a tiny fraction of $k$. The corresponding upper and lower bars' resonant frequency lines split around the nominal 7Hz line, as $\overline{\omega}_0, \underline{\omega}_0 = 7.0 \pm 1.4 \times 10^{-10}$ Hz, not observable experimentally. Likewise, the upper and lower bars' d.c.

amplitudes $(I_1 - I_2)\,\phi_{12}/\kappa$ give only slightly different measured average angles $\overline{\theta}_0, -\underline{\theta}_0 = (-5.656 \times 10^{-10} \pm 2.2 \times 10^{-20})^\circ$. These numbers definitely support the first-order measurement formulas (40) and (41) above.

Before going on, we pause to emphasize an almost obvious, but vital, point about our measurement model. By formula (9), the contribution to the gravity gradient $\phi_{ab}$ from each source $i$ is proportional to its mass $m_i$. The constant of proportionality is a known function of the coordinate geometry of the source. This means that, if we scan the surface of a volume containing some source masses, measuring $\theta_0$ at several points on that surface, then in principle, we can solve for the source mass $m_i$ at each point inside the volume.

**Second-order torque.** We continue our Taylor expansion of the gravity field. The next piece of the torque on the bar due to gravity is that of second-order. We rotate again to the body frame (primed coordinates):

$$\begin{aligned}\tau_a'^{(2)} &= -\frac{1}{2}\int \epsilon_{abc} r_b' \cdot (\phi_{cde}' r_d' r_e') \cdot\ \sigma(\mathbf{r}')dV' \\ &= -\frac{1}{2}\epsilon_{abc}\phi_{cde}' \int r_b' r_d' r_e'\ \sigma(\mathbf{r}')dV'.\end{aligned}$$

or, rotating the three terms, to first order in $\theta$ we find that

$$\begin{aligned}\tau_a^{(2)} &= -\frac{1}{2}\epsilon_{abc}\left(\phi_{cde} + \left[\epsilon_{3cg}\phi_{gde} + \epsilon_{3dh}\phi_{che} + \epsilon_{3ei}\phi_{cdi}\right]\theta\right) \\ &\qquad \times \left(\int r_b' r_d' r_e'\ \sigma(\mathbf{r}')dV'\right). \end{aligned} \tag{42}$$

Much as before, this expression separates into third derivatives $\phi_{cde}$ of the local gravity potential field, and the third-order moments $I_{bcd}$ of the given sensing bar.

However, for our symmetric bar, the reader can verify, without doing the integrals $\sigma \int r_b' r_d' r_e' dV'$, that all of these third-order moments are zero. The principal axes for the body frame are just the three axes of symmetry of the bar. Our bar has uniform mass density $\sigma(\mathbf{r}') = \sigma$. Therefore, every volume element such as $xyz\ dV$ can be paired with another volume element $-xyz\ dV$. The same is true for all the other integrands, such as $x^3$, $x^2y$, etc.

In general, for the same reasons, we conclude that all moments of odd degree $(1, 3, 5, \ldots)$ are zero for our symmetric bar with pivot

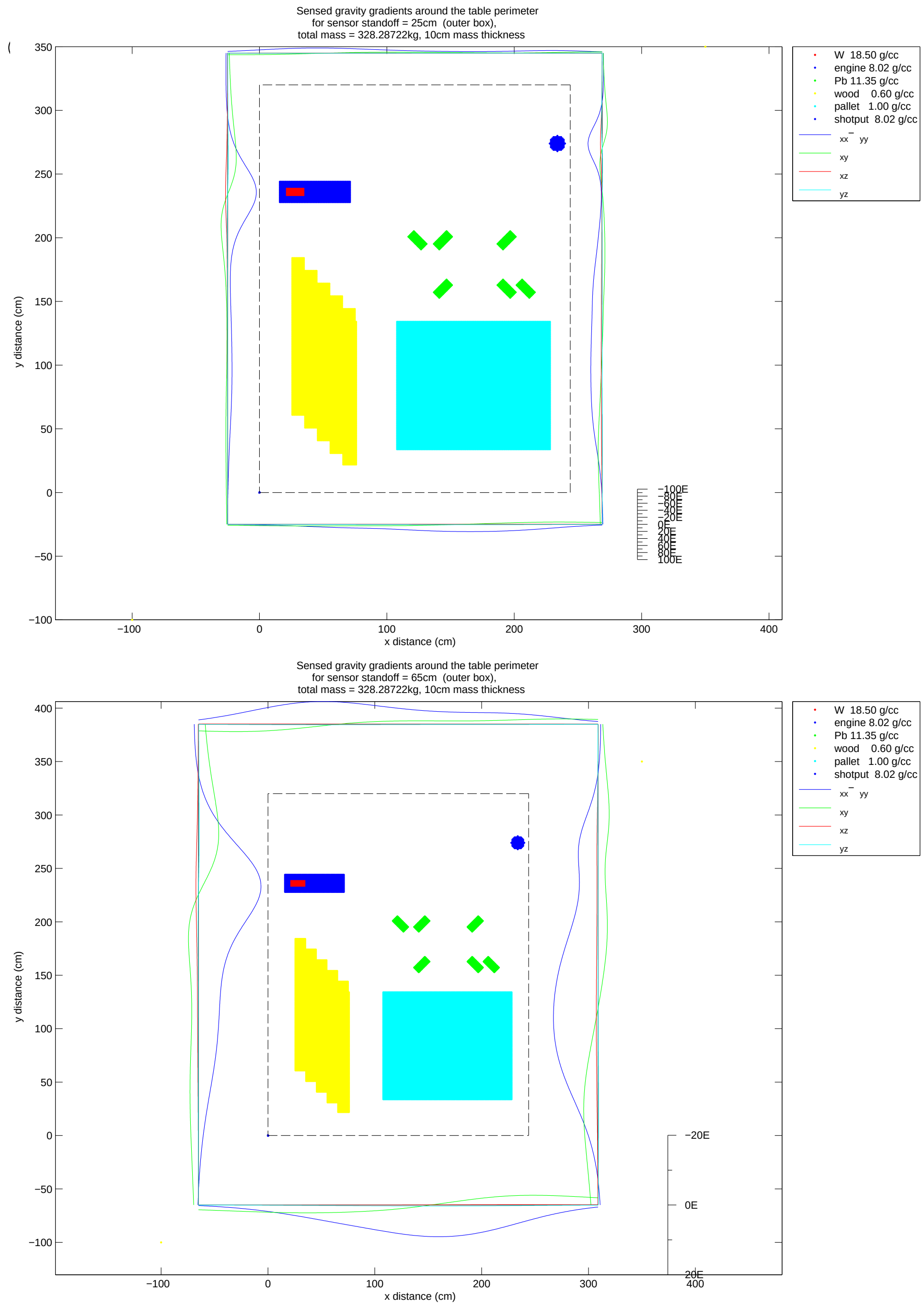


Figure 2: Tidal tensor components of gravity for a close-range tabletop scan for standoff distances 25cm (top), 65cm (bottom). The tabletop boundary is the black dashed rectangle. The sensor scan path is the solid black rectangle outside the tabletop. The corresponding gradient curves are overlaid using the scan rectangle as the plot axis at zero level. The scale for the curves is set so they fill the standoff region in each case: $\pm 100$E (top), $\pm 20$E (bottom).

at its CM. Moreover, the moments of even degree that contain an odd power (such as $xy$ or $x^3y^1z^0$) must be zero too, due to 3-axis symmetry. Small non-zero odd moments could result if the pivot is displaced a small distance $d$ from the bar's center of mass, but for now we neglect this for the higher moments. The first moment could easily be adjusted from 0, and Appendix A gives the adjustment for the second moments of inertia.

**Third-order torque.** The reader is now equipped, in principle, to write out the third, fifth, seventh and higher non-zero torque terms, as needed. We conclude with the third-order term, rotating to the body frame (primed coordinates):

$$\begin{aligned}\tau_a'^{(3)} &= -\frac{1}{3!}\int \epsilon_{abc} r_b' \cdot \left(\phi'_{cdef} r_d' r_e' r_f'\right) \cdot \ \sigma(\mathbf{r}')dV' \\ &= -\frac{1}{3!}\epsilon_{abc}\phi'_{cdef}\int r_b' r_d' r_e' r_f' \ \sigma(\mathbf{r}')dV'. \end{aligned} \tag{43}$$

or, rotating $\phi'_{cdef}$ four ways, to first order in $\theta$ we find that

$$\begin{aligned}\tau_a^{(3)} &= -\frac{1}{3!}\epsilon_{abc}\left(\phi_{cdef} + \left[\epsilon_{3cg}\phi_{gdef} + \epsilon_{3dh}\phi_{chef} + \epsilon_{3ei}\phi_{cdif} + \epsilon_{3fj}\phi_{cdej}\right]\theta\right) \\ &\quad \times \left(\int r_b' r_d' r_e' r_f' \ \sigma(\mathbf{r}')dV'\right). \end{aligned} \tag{44}$$

As before, this expression separates, now into fourth derivatives $\phi_{cdef}$ of the local gravity potential field sensed at the bar's CM, and the fourth moments $I_{bdef} = \int r_b' r_d' r_e' r_f' \ \sigma(\mathbf{r}')dV'$ of the given sensing bar with respect to its CM. The coefficients of $\theta$ become part of the effective spring constant $\kappa$, so we do not consider them further. The terms from $\phi_{cdef}$ become part of $\theta_0$, the equilibrium angle of the rotor bar.

Many of these terms drop out. The moment integrals (at the end of the right side above) with an odd power of any $r_k'$ coordinate are zero. We want the $z$ component of torque, so $a = 3$. Then $\epsilon_{abc}$ is zero unless $bc = 12$ or 21. For the $bdef$ indices, the even moments leave us these choices (out of $3^4$):

| moment | $\epsilon_{3bc}$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $x^4$ | + | 1 | 2 | 1 | 1 | 1 |
| $x^2y^2$ | + | 1 | 2 | 1 | 2 | 2 |
| | + | 1 | 2 | 2 | 1 | 2 |
| | + | 1 | 2 | 2 | 2 | 1 |
| $x^2z^2$ | + | 1 | 2 | 1 | 3 | 3 |
| | + | 1 | 2 | 3 | 1 | 3 |
| | + | 1 | 2 | 3 | 3 | 1 |
| $y^4$ | - | 2 | 1 | 2 | 2 | 2 |
| $x^2y^2$ | - | 2 | 1 | 2 | 1 | 1 |
| | - | 2 | 1 | 1 | 2 | 1 |
| | - | 2 | 1 | 1 | 1 | 2 |
| $y^2z^2$ | - | 2 | 1 | 2 | 3 | 3 |
| | - | 2 | 1 | 3 | 2 | 3 |
| | - | 2 | 1 | 3 | 3 | 2 |

This list shows the indices of the $\phi'_{cdef}$ derivatives to be summed for each moment $\int r'_b r'_d r'_e r'_f \; dV$. Keep in mind the partial derivatives $\phi'_{cdef}$ are the same when the indices are permuted. Thus $\phi'_{xyyy} \equiv \phi'_{1222}$ and $\phi'_{xyxx}$ occur four times each, and $\phi'_{xyzz}$ occurs six times. Also, because $b \neq 3$ in our formula for $a = 3$, there is no $z^4$ moment. Therefore, we can sum up the first term of (44) as:

$$\begin{aligned} -\frac{1}{3!}\epsilon_{abc}\phi_{cdef} \cdot \int r'_b r'_d r'_e r'_f \; \sigma(\mathbf{r}')dV' &= -\frac{1}{3!}\left(\left[I(x^4) - 3I(x^2y^2)\right]\phi_{xyxx}\right. \\ &\quad - \left[I(y^4) - 3I(x^2y^2)\right]\phi_{xyyy} \\ &\quad \left. +3\left[I(x^2z^2) - I(y^2z^2)\right]\phi_{xyzz}\right). \end{aligned}$$

where from (15), these 4th derivatives simplify as:

$$\begin{aligned} \phi_{xyxx} &= \frac{15Gm}{s^5}\left(\frac{s_x}{s}\frac{s_y}{s}\right)\left(3 - 7\left(\frac{s_x}{s}\right)^2\right) \\ \phi_{xyyy} &= \frac{15Gm}{s^5}\left(\frac{s_x}{s}\frac{s_y}{s}\right)\left(3 - 7\left(\frac{s_y}{s}\right)^2\right) \\ \phi_{xyzz} &= \frac{15Gm}{s^5}\left(\frac{s_x}{s}\frac{s_y}{s}\right)\left(1 - 7\left(\frac{s_z}{s}\right)^2\right). \end{aligned} \tag{45}$$

Note these three derivatives sum to 0, as they should. This follows from operating by $\frac{\partial^2}{\partial x \partial y}$ on both sides of the Laplace equation. More

4th derivatives:

$$
\begin{aligned}
\phi_{xxyy} &= \frac{15Gm}{s^5}\left(-\frac{1}{5}+\frac{s_x^2}{s^2}+\frac{s_y^2}{s^2}-7\left(\frac{s_x s_y}{s^2}\right)^2\right)\\
\phi_{xxzz} &= \frac{15Gm}{s^5}\left(-\frac{1}{5}+\frac{s_x^2}{s^2}+\frac{s_z^2}{s^2}-7\left(\frac{s_x s_z}{s^2}\right)^2\right)\\
\phi_{yyzz} &= \frac{15Gm}{s^5}\left(-\frac{1}{5}+\frac{s_y^2}{s^2}+\frac{s_z^2}{s^2}-7\left(\frac{s_y s_z}{s^2}\right)^2\right)\\
\phi_{xxxx} &= \frac{15Gm}{s^5}\left(-\frac{3}{5}+\frac{6s_x^2}{s^2}-7\frac{s_x^4}{s^4}\right)\\
\phi_{yyyy} &= \frac{15Gm}{s^5}\left(-\frac{3}{5}+\frac{6s_y^2}{s^2}-7\frac{s_y^4}{s^4}\right)\\
\phi_{zzzz} &= \frac{15Gm}{s^5}\left(-\frac{3}{5}+\frac{6s_z^2}{s^2}-7\frac{s_z^4}{s^4}\right). \qquad (46)
\end{aligned}
$$

Formulas for the even-power moments of a twin-diamond rotor are given in Appendix B. The exact formulas for $\phi'_{cdef}$ after any rotation are given in Appendix C.

Consider the third-order torque term. Evaluating the formula (65) in Appendix C for the first bar's space frame angle $\alpha = 0$, and for small bar displacements $\theta$,

$$
\begin{aligned}
\overline{\phi}'_{1211} &= \overline{\phi}_{1211} + \left(3\overline{\phi}_{1212} - 2\overline{\phi}_{1111}\right)\theta\\
\overline{\phi}'_{1222} &= \overline{\phi}_{1222} - \left(3\overline{\phi}_{1212} - 2\overline{\phi}_{2222}\right)\theta\\
\overline{\phi}'_{1233} &= \overline{\phi}_{1233} - \left(\overline{\phi}_{1133} - \overline{\phi}_{2233}\right)\theta. \qquad (47)
\end{aligned}
$$

For the other bar at space-frame angle $\alpha = \pi/2$,

$$
\begin{aligned}
\underline{\phi}'_{1211} &= -\underline{\phi}_{1222} + \left(3\underline{\phi}_{1212} - 2\overline{\phi}_{2222}\right)\theta\\
\underline{\phi}'_{1222} &= -\underline{\phi}_{1211} - \left(3\underline{\phi}_{1212} - 2\overline{\phi}_{1111}\right)\theta\\
\underline{\phi}'_{1233} &= -\underline{\phi}_{1233} + \left(\underline{\phi}_{1133} - \underline{\phi}_{2233}\right)\theta. \qquad (48)
\end{aligned}
$$

Comparing formulas, we see that the rotation of the lower bar by $\alpha = \pi/2$ not only reverses the signs, but also exchanges the $xyxx$ and $xyyy$ derivatives, which have different moment coefficients. The third-order torque on the lower bar is not simply the negative of that on the upper bar, as it was for the first-order torque.

Finally, our bottom line. The harmonic oscillator equation (38) still holds. The effective spring constant $\kappa$ includes more coefficients of $\theta$, but is measurable just as $\kappa = I_3\omega_0^2$. (We keep the formulas to first order in $\Delta k$.) For the total torque up to order 3, up to a scale factor we can measure

$$\boxed{\begin{aligned}\left(\overline{\theta}_0 - \underline{\theta}_0\right) &= [I_1 - I_2]\left[\overline{\phi}_{xy} + \underline{\phi}_{xy}\right] \\ &\quad -\tfrac{1}{3!}\Big\{[I_{x^4} - 3I_{x^2y^2}]\left[\overline{\phi}_{xyxx} + \underline{\phi}_{xyyy}\right] \\ &\qquad - [I_{y^4} - 3I_{x^2y^2}]\left[\overline{\phi}_{xyyy} + \underline{\phi}_{xyxx}\right] \\ &\qquad + 3\,[I_{x^2z^2} - I_{y^2z^2}]\left[\overline{\phi}_{xyzz} + \underline{\phi}_{xyzz}\right]\Big\}\,.\end{aligned}} \tag{49}$$

We have distinguished the partial derivatives $\overline{\phi}$, $\underline{\phi}$ of the gravity field at the upper and lower bars' centers, respectively oriented at $\alpha = 0, \pi/2$. They are evaluated at the bars' CMs, about 8cm vertically apart. A similar formula holds for the bars oriented at $\alpha = \pi/4, 3\pi/4$ (see the complete derivative formula at the end of Appendix C). The Taylor expansion for each bar uses the incremental distances from its CM, ranging about $\pm$4cm in the horizontal plane. The source masses are assumed to be as close as about 65cm. So we do not assume the partial derivatives of the gravity potential at the upper CM equal those of the potential at the lower bar's CM. (They are equal only for the symmetric case when every point-source mass lies in the plane halfway between the two bars.)

Again, the derivatives are all proportional to the point-source mass $m_i$. So when we take the sum of the right hand side over all source masses, we have a linear combination of those masses. From the measurements, we can recover the masses.

Figure 3 shows that the gradient (second-order) angle-difference between the two bars part is much greater than the fourth-order part, especially for the 65cm standoff. We have magnified the 4th-order by two orders of magnitude (20dB) to make it easy to see. The total angle difference is the sum of the two parts. (Recall the third-order part was 0, due to symmetries of the rotor bar.) The figure suggests that we may be able to recover the masses from

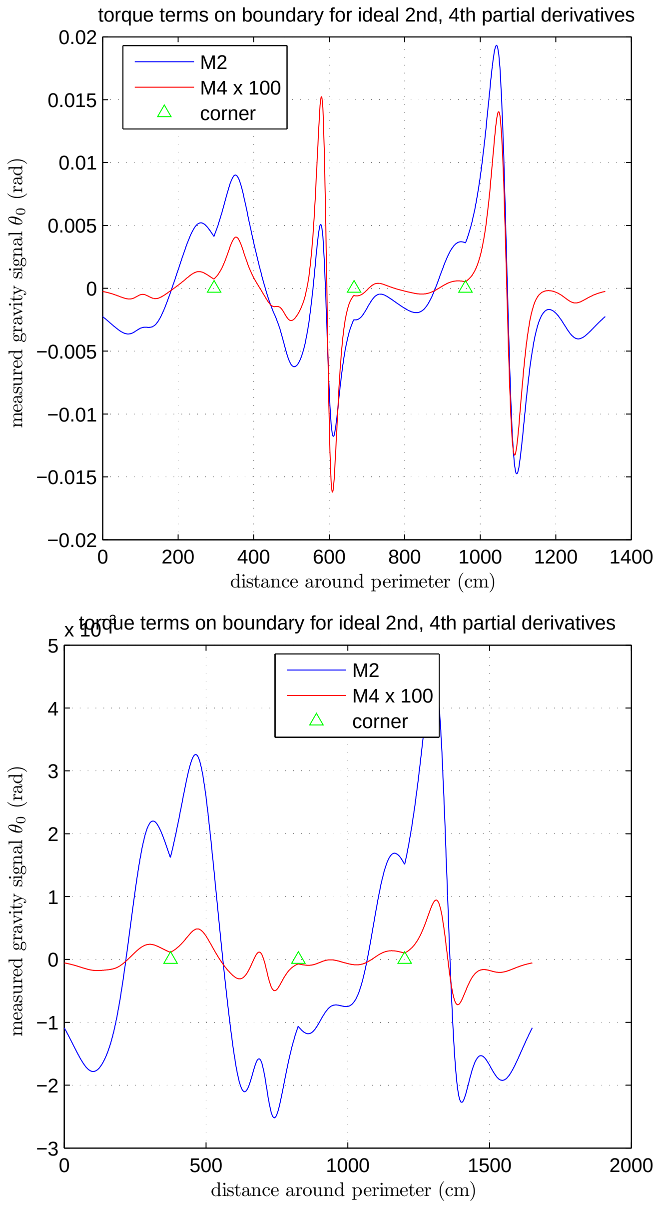


Figure 3: Measurable gravity signal (difference of two orthogonal sensor bars' displacement angles $\theta$) from a close-range tabletop scan for standoff distances 25cm (top), 65cm (bottom). The 2nd-order (M2) component corresponds to second derivatives $\phi_{yy} - \phi_{xx}$ of the gravity potential $\phi$. The 4th-order (M4) component corresponds to the 4th derivatives and is magnified by 100×. All odd components are 0 due to bar symmetry.

the 2nd-order part only, given that the intrinsic instrument noise is much greater than the 4th-order part. But for very low noise and closer standoffs, we may sometimes do better if we add the 4th-order contributions in our forward model as in the formula above. They appear to make a small difference.

## 5 Conclusion

This paper uses tensors to facilitate Newtonian gravity calculations. We have focused on the basic mechanics ("gravitostatics") for short range gravity gradiometry of static mass distributions, where the distance of the masses is not much greater than the size and spacing of the pair of sensing bars. Noises and errors are addressed in depth in the references.

Complete formulas were given for computing the 2nd and 4th order terms of the torque on a sensor bar, due to gravity gradients. (The odd-order terms are zero due to bar symmetries.) This gave us an equation of motion for the deflection angle of the bar on its flexure pivot, in the bar's body frame. Given the bar's moments of inertia, we then evaluated the angle difference between a pair of ideal twin-diamond sensor bars at static equilibrium.

For each sensor position, this measured average angle difference is a linear combination of the unknown source masses. The combining coefficients are a function of the gradients of the gravity field and the bars' moments of inertia. Therefore, when the sensor scans the source masses, they can be recovered by optimization methods.

## 6 Acknowledgements

I would like to thank Kieran Carroll, David Hatch, and M. Vol Moody for discussions on this topic, and for introducing me to formula (35) [3][4].

## 7 References

[1] James B. Hartle, *Gravity: An introduction to Einstein's general relativity*, Addison-Wesley, 2003.

[2] Herbert Goldstein, Charles Poole and John Safko, *Classical Mechanics*, 4th ed., Addison-Wesley, 2002.

[3] Kieran Carroll, *Torque on a body due to the local gravity gradient tensor*, Gedex Inc., June 19, 2009.
[4] M. Vol Moody, Ho Jung Paik, Edgar R. Canavan, "Principle and performance of a superconducting angular accelerometer," Review of Scientific Instruments, vol. 74, No. 3, Mar. 2003, pp. 1310-1318.
[5] M. Vol Moody, Ho Jung Paik, Edgar R. Canavan, "Three-axis superconducting gravity gradiometer for sensitive gravity experiments," Review of Scientific Instruments, vol. 73, No. 11, Nov. 2002, pp. 3957-3974.
[6] Gabriela I. Gonzalez, *Brownian motion of a torsion pendulum damped by internal friction,* PhD thesis (Peter Saulson, advisor) Syracuse Univ., 1995.

# Appendix A. **Second moments of inertia of the twin-diamond rotor**

Consider an idealized torsion bar or balance beam or rotor (our gravity sensing element) made of two identical squares of metal, as drawn from a top view in Figure 1. The squares are $2a$ thick and have sides of length $2b$. A typical bar has thickness less than the square side: $a < b$. The squares are both oriented as diamonds (at a 45° tilt) and are rigidly connected corner-to-corner by a short neck of metal of length $2 \cdot c\sqrt{2}$, whose mass we neglect. The two-square bar assembly turns about the $z$-axis on a flexure pivot at its center of mass (CM) mid-neck. The volume of a square is $V = 2a \cdot (2b)^2 = 8ab^2$. For a bar of density $\sigma$, the mass of a square is $M = V\sigma$. (A superconducting alloy of niobium (8.57g/cc) and titanium (4.5g/cc) has a lower density or specific gravity, but for this example we have assumed pure niobium, $\sigma = 8.57$g/cc.)

We take the axes of symmetry for this bar, its principal axes. Then all cross-moments (such as the $xy$ moment) are 0. The bar lies parallel to the $x$-$y$ plane, with the origin at its center of mass. Its uniform thickness means its $z$-coordinates vary between the bounds $\pm a$. The $x$-axis runs through the square diagonals and the connecting neck, and the $y$-axis is normal to this in the plane of the squares.

To compute the (second) moments of inertia, we will use the parallel axis theorem to move our axis of rotation from the CM to a new axis parallel to it. The theorem states that the moments about the new axis equal the original moments through the CM, plus the moments about the new axis of a point mass located at the CM. So by adding an easy calculation for the moment of one point, we can parallel-translate the axis from the CM to any new point.

To begin calculating, we compute the moments for one square alone, for the axes through its center. First we find the integrals

$$\begin{aligned}
\sigma \int z^2 dV &= \sigma \int_{-a}^{a} z^2 dz \int dx dy = \frac{8}{3} a^3 b^2 \sigma = \frac{M}{3} a^2 \\
\sigma \int y^2 dV &= \sigma \int x^2 dV \\
&= \sigma 2a \int x^2 dx dy = 4 \cdot \sigma 2a \int_0^{b\sqrt{2}} x^2 dx \int_0^{b\sqrt{2}-x} dy = \frac{M}{3} b^2,
\end{aligned}$$

where we quadrupled the moment of a quarter square. So the principal moments for the square are

$$\begin{aligned}
I_2^s &= I_1^s \\
&= \sigma \int \left(y^2 + z^2\right) dV = \frac{M}{3} \left(a^2 + b^2\right) \\
I_3^s &= \sigma \int \left(x^2 + y^2\right) dV = \frac{M}{3} \left(b^2 + b^2\right).
\end{aligned}$$

For the two-square configuration, by symmetry the moments are just twice those of a single square. The CM of each square shifts to center along the $x$-axis by $\pm(c+b)\sqrt{2}$, so we must add $2M(c+b)^2$ to the moments of one metal square parallelopiped about the $y$ and $z$ axes. So the principal moments for the two-square bar are

$$I_1 = \frac{2M}{3} \left(a^2 + b^2\right) \tag{50}$$

$$I_2 = \frac{2M}{3} \left(a^2 + b^2 + 6(c+b)^2\right) \tag{51}$$

$$I_3 = \frac{2M}{3} \left(2b^2 + 6(c+b)^2\right). \tag{52}$$

Finally, suppose the pivot is actually translated a (small) distance $d$ from the CM of the two squares, say in the $\pm y$ direction. The $y$-axis remains the same, but the $x$ and $z$ axes are translated parallel to themselves so that they pass through the new pivot. We can use the parallel axis theorem again. We just add twice $Md^2$ to the moments *not* about the $y$-axis (i.e., to the moments about the $x$ and

$z$ axes):

$$I_1 = \frac{2M}{3}\left(a^2+b^2+3d^2\right)$$

$$I_2 = \frac{2M}{3}\left(a^2+b^2+6(c+b)^2\right)$$

$$I_3 = \frac{2M}{3}\left(2b^2+6(c+b)^2+3d^2\right).$$

## Appendix B. **Fourth moments of the twin-diamond rotor**

For fourth moments, a parallel axis formula may be possible, as for the second moments. But we have not worked one out. So instead we integrate one diamond (a right parallelopiped) directly from the origin at the CM of the two conjoined diamonds.

*Notes:* We multiply by 2 in every case, to get the total moment for both squares, tilted as diamonds. Often we only have to integrate the top triangle of the square above the $x$-axis, and we multiply this by 2 to include the bottom half. Still another factor of $2a$ comes with the $z$-free moments: we get a factor $\int dz = 2a$. Since these moments are integrals of a product of four variables $r_c r_c r_d r_e$, but there are of course only three to choose from, $x, y, z$, at least one variable occurs twice in the product. Only the six moments below, comprising products of even powers of coordinates, are nonzero. The results are always polynomials of degree 7 in the three diamond parameters $a, b, c$. As in Appendix A, the mass of one square is $M = \sigma V = \sigma 8ab^2$. When we take out $M$, what is left has degree 4, giving the correct dimensions for a fourth moment.

$$I(z^4) = 2\sigma\int z^4 dV = 2\cdot(2b)^2\sigma\int_{-a}^{a} z^4 dz = \frac{16}{5}a^5b^2\sigma = \frac{2M}{5}a^4. \quad (53)$$

$$\begin{aligned} I(y^4) = 2\sigma\int y^4 dV &= 8a\sigma\int_0^{b\sqrt{2}} y^4\int_{c\sqrt{2}+y}^{(c+2b)\sqrt{2}-y} dxdy \\ &= 8a\sigma\int_0^{b\sqrt{2}} y^4\left(2b\sqrt{2}-2y\right)dy \\ &= 16a\sigma\left(\frac{1}{5}b\sqrt{2}y^5-\frac{1}{6}y^6\right)\Big|_0^{b\sqrt{2}} = \frac{8M}{15}b^4. \end{aligned} \quad (54)$$

$$
\begin{aligned}
I(x^4) = 2\sigma \int x^4 dV &= 8a\sigma \left( \int_{c\sqrt{2}}^{(c+b)\sqrt{2}} x^4 \int_0^{x-c\sqrt{2}} dydx + \int_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} x^4 \int_0^{(c+2b)\sqrt{2}-x} dydx \right) \\
&= 8a\sigma \left( \frac{1}{6}x^6 - \frac{1}{5}c\sqrt{2}x^5 \right)\Big|_{c\sqrt{2}}^{(c+b)\sqrt{2}} - 8a\sigma \left( \frac{1}{6}x^6 - \frac{1}{5}(c+2b)\sqrt{2}x^5 \right)\Big|_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} \\
&= \frac{32}{15}a\sigma \left( c^6 - 2(c+b)^6 + (c+2b)^6 \right) \\
&= 8M \left( (c+b)^4 + b^2(c+b)^2 + \frac{1}{15}b^4 \right). \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
I(x^2y^2) = 2\sigma \int x^2y^2 dV &= 8a\sigma \left( \int_{c\sqrt{2}}^{(c+b)\sqrt{2}} x^2 \int_0^{x-c\sqrt{2}} y^2 dydx + \int_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} x^2 \int_0^{(c+2b)\sqrt{2}-x} y^2 dydx \right) \\
&= 8a\sigma \left( \int_{c\sqrt{2}}^{(c+b)\sqrt{2}} x^2 \left( x - c\sqrt{2} \right)^3 /3dx + \int_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} x^2 \left( (c+2b)\sqrt{2} - x \right)^3 /3dx \right) \\
&= 8a\sigma \left( \int_0^{b\sqrt{2}} (c\sqrt{2}+x)^2 x^3/3dx + \int_0^{b\sqrt{2}} ((c+2b)\sqrt{2}+x)^2 x^3/3dx \right) \\
&= \frac{4}{45}Mb^2 \left( 64b^2 + 54cb + 15c^2 \right). \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
I(x^2z^2) = 2\sigma \int x^2z^2 dV &= \frac{8}{3}a^3\sigma \left( \int_{c\sqrt{2}}^{(c+b)\sqrt{2}} x^2 \int_0^{x-c\sqrt{2}} dydx + \int_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} x^2 \int_0^{(c+2b)\sqrt{2}-x} dydx \right) \\
&= \frac{8}{3}a^3\sigma \left\{ \left( \frac{1}{4}x^4 - \frac{1}{3}c\sqrt{2}x^3 \right)\Big|_{c\sqrt{2}}^{(c+b)\sqrt{2}} - \left( \frac{1}{4}x^4 - \frac{1}{3}(c+2b)\sqrt{2}x^3 \right)\Big|_{(c+b)\sqrt{2}}^{(c+2b)\sqrt{2}} \right\} \\
&= \frac{8}{9}a^3\sigma \left( c^4 - 2(c+b)^4 + (c+2b)^4 \right) = \frac{2}{9}Ma^2 \left( b^2 + 6(c+b)^2 \right). \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
I(y^2z^2) = 2\sigma \int y^2z^2 dV &= \frac{8}{3}a^3\sigma \int_0^{b\sqrt{2}} y^2 \int_{c\sqrt{2}+y}^{(c+2b)\sqrt{2}-y} dxdy \\
&= \frac{8}{3}a^3\sigma \int_0^{b\sqrt{2}} y^2 \left( 2b\sqrt{2} - 2y \right) dy \\
&= \frac{16}{3}a^3\sigma \left( \frac{1}{3}b\sqrt{2}y^3 - \frac{1}{4}y^4 \right)\Big|_0^{b\sqrt{2}} = \frac{2}{9}Ma^2b^2. \qquad (58)
\end{aligned}
$$

These formulas were checked using Matlab's polyint and polyval functions. Checks of special cases can also be made. For example, we can take the limit as each diamond shrinks to a point mass. We keep both masses $M$ and their centers of mass (CM) constant. The CM for each diamond keeps its original value, CM $= \pm x_0$, where $x_0 = (c_0 + b_0)\sqrt{2}$. Then let the spacing $c\sqrt{2} \to x_0$, as $a, b \to 0$. Only $I(x^4) = 8M(c_0 + b_0)^4 = 2Mx_0^4$ is non-zero.

*Example:* Let $a = 1.27$cm, $b = 4.1275$cm, $c = 0.15875$cm. Then the (second, principal) moments of inertia about the $x, y, z$ axes are, in units of g-cm$^2$, respectively:

$$I_a^{(2)} = \begin{bmatrix} 18442.365 \\ 127451.698 \\ 142704.033 \end{bmatrix},$$

and $I_2 - I_1 = 109009.33$. We can arrange the non-zero fourth moments (a rank-4 tensor) as a symmetric rank-2 tensor of "even moments", in units of g-cm$^4$:

$$I_{ab}^{(4)\text{even}} \equiv \begin{bmatrix} I(x^4) & I(x^2y^2) & I(x^2z^2) \\ I(y^2x^2) & I(y^4) & I(y^2z^2) \\ I(z^2x^2) & I(z^2y^2) & I(z^4) \end{bmatrix}$$
$$= \begin{bmatrix} 7949260.107 & 2529531.703 & 67664.748 \\ 2529531.703 & 229612.618 & 9057.697 \\ 67664.748 & 9057.697 & 1543.560 \end{bmatrix}.$$

Appendix C. **Fourth derivatives $\phi'_{cdef}$ of the gravity potential in the body frame** Let the body frame be the space frame rotated by $\theta$ about the $z$-axis. Both frames have the CM of the bar as their origin. We will use corresponding indices

$$\begin{array}{cccc} c & d & e & f \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ g & h & i & j \end{array}$$

each taking values 1, 2, 3. To change coordinates, apply the (passive) $z$-rotation $R$ (see (8)) along each axis of the rank-4 tensor, i.e. write out each derivative $\phi'_{cdef} = R_{fj}R_{ei}R_{dh}R_{cg}\phi_{ghij}$ for $c = 1, d = 2$ and $e = f = 1, 2, 3$, those which we need in the term (43) from which we measure the equilibrium bar angle $\theta_0$. (The order does not matter since each rotation affects a different dimension of the tensor $\phi$.) To begin, the first rotation for $c = 1$ dots row 1 of $R$ for $g = 1, 2, 3$ (really just $g = 1, 2$) with the $g$ dimension of $\phi_{ghij}$ to give

$$\phi'_{1hij} \equiv R_{1g}\phi_{ghij} = c_\theta\phi_{1hij} + s_\theta\phi_{2hij}.$$

We label this intermediate result as $\phi'_{1hij}$. Then the next rotation for $d = 2$ uses row 2 of $R$ for $h = 1, 2, 3$ (really just $h = 1, 2$) and gives us

$$\begin{aligned}\phi'_{12ij} \equiv R_{2h}\phi'_{1hij} &= -s_\theta c_\theta \phi_{11ij} - s_\theta^2 \phi_{21ij} + c_\theta^2 \phi_{12ij} + c_\theta s_\theta \phi_{22ij} \\ &= c_{2\theta}\phi_{12ij} + \tfrac{1}{2}s_{2\theta}\left(\phi_{22ij} - \phi_{11ij}\right), \end{aligned} \tag{59}$$

where for brevity we put $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, $c_{2\theta} = \cos 2\theta$, etc. All three derivatives can now be computed from this intermediate step (59). We leave the next two rotations for the three cases $e = f = 1, 2, 3$ as an exercise for the reader (you will like $e = f = 3$), and jump to the answers. After collecting terms and condensing with trigonometric identities, we find

$$\begin{aligned}\phi'_{1211} &= \tfrac{1}{2}s_{2\theta}\left(s_\theta^2\phi_{2222} - c_\theta^2\phi_{1111}\right) \\ &\quad + \left(c_\theta c_{3\theta}\right)\phi_{1112} + \left(s_\theta s_{3\theta}\right)\phi_{1222} \\ &\quad + \tfrac{3}{4}s_{4\theta}\phi_{1122} \\ \\ \phi'_{1222} &= \tfrac{1}{2}s_{2\theta}\left(c_\theta^2\phi_{2222} - s_\theta^2\phi_{1111}\right) \\ &\quad + \left(s_\theta s_{3\theta}\right)\phi_{1112} + \left(c_\theta c_{3\theta}\right)\phi_{1222} \\ &\quad - \tfrac{3}{4}s_{4\theta}\phi_{1122} \\ \\ \phi'_{1233} &= c_{2\theta}\phi_{1233} + \tfrac{1}{2}s_{2\theta}\left(\phi_{2233} - \phi_{1133}\right).\end{aligned}$$

Some trigonometric polishing yields

$$\begin{aligned}\phi'_{1211} &= \left(\tfrac{1}{2}s_{2\theta} - \tfrac{1}{4}s_{4\theta}\right)\phi_{2222} - \left(\tfrac{1}{2}s_{2\theta} + \tfrac{1}{4}s_{4\theta}\right)\phi_{1111} \\ &\quad + \left(\tfrac{1}{2}c_{2\theta} + \tfrac{1}{2}c_{4\theta}\right)\phi_{1112} + \left(\tfrac{1}{2}c_{2\theta} - \tfrac{1}{2}c_{4\theta}\right)\phi_{1222} \\ &\quad + \tfrac{3}{4}s_{4\theta}\phi_{1122} \end{aligned} \tag{60}$$

$$\begin{aligned}\phi'_{1222} &= \left(\tfrac{1}{2}s_{2\theta} + \tfrac{1}{4}s_{4\theta}\right)\phi_{2222} - \left(\tfrac{1}{2}s_{2\theta} - \tfrac{1}{4}s_{4\theta}\right)\phi_{1111} \\ &\quad + \left(\tfrac{1}{2}c_{2\theta} - \tfrac{1}{2}c_{4\theta}\right)\phi_{1112} + \left(\tfrac{1}{2}c_{2\theta} + \tfrac{1}{2}c_{4\theta}\right)\phi_{1222} \\ &\quad - \tfrac{3}{4}s_{4\theta}\phi_{1122} \end{aligned} \tag{61}$$

$$\phi'_{1233} = c_{2\theta}\phi_{1233} + \tfrac{1}{2}s_{2\theta}\left(\phi_{2233} - \phi_{1133}\right), \tag{62}$$

Last, we substitute $\theta \leftarrow \theta'(t) = \alpha + \theta(t)$, where $\alpha$ is a large fixed angle between the space frame and the body frame at rest. This is the small-$\theta$ approximation for the gravity-field derivatives in the body frame:

$$\begin{aligned}\phi'_{1211} = & \left(\tfrac{1}{2}s_{2\alpha} - \tfrac{1}{4}s_{4\alpha} + (c_{2\alpha} - c_{4\alpha})\,\theta\right)\phi_{2222} \\ & - \left(\tfrac{1}{2}s_{2\alpha} + \tfrac{1}{4}s_{4\alpha} + (c_{2\alpha} + c_{4\alpha})\,\theta\right)\phi_{1111} \\ & + \left(\tfrac{1}{2}c_{2\alpha} + \tfrac{1}{2}c_{4\alpha} - (s_{2\alpha} + 2s_{4\alpha})\,\theta\right)\phi_{1112} \\ & + \left(\tfrac{1}{2}c_{2\alpha} - \tfrac{1}{2}c_{4\alpha} - (s_{2\alpha} - 2s_{4\alpha})\,\theta\right)\phi_{1222} \\ & + \tfrac{3}{4}\left(s_{4\alpha} + 4c_{4\alpha}\theta\right)\phi_{1122} \end{aligned} \quad (63)$$

$$\begin{aligned}\phi'_{1222} = & \left(\tfrac{1}{2}s_{2\alpha} + \tfrac{1}{4}s_{4\alpha} + (c_{2\alpha} + c_{4\alpha})\,\theta\right)\phi_{2222} \\ & - \left(\tfrac{1}{2}s_{2\alpha} - \tfrac{1}{4}s_{4\alpha} + (c_{2\alpha} - c_{4\alpha})\,\theta\right)\phi_{1111} \\ & + \left(\tfrac{1}{2}c_{2\alpha} - \tfrac{1}{2}c_{4\alpha} - (s_{2\alpha} - 2s_{4\alpha})\,\theta\right)\phi_{1112} \\ & + \left(\tfrac{1}{2}c_{2\alpha} + \tfrac{1}{2}c_{4\alpha} - (s_{2\alpha} + 2s_{4\alpha})\,\theta\right)\phi_{1222} \\ & - \tfrac{3}{4}\left(s_{4\alpha} + 4c_{4\alpha}\theta\right)\phi_{1122} \end{aligned} \quad (64)$$

$$\begin{aligned}\phi'_{1233} = & \left(c_{2\alpha} - 2s_{2\alpha}\theta\right)\phi_{1233} \\ & + \left(\tfrac{1}{2}s_{2\alpha} + c_{2\alpha}\theta\right)\left(\phi_{2233} - \phi_{1133}\right). \end{aligned} \quad (65)$$

For convenience, here is a short table of values for some orientation angles of interest:

| $\alpha$ | $c_{2\alpha}$ | $s_{2\alpha}$ | $c_{4\alpha}$ | $s_{4\alpha}$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 |
| $\pi/4$ | 0 | 1 | -1 | 0 |
| $\pi/2$ | -1 | 0 | 1 | 0 |
| $3\pi/4$ | 0 | -1 | -1 | 0 |