On the Use of Shor States for the $[7, 1, 3]$ Quantum Error Correcting Code

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We explore the effect of Shor state construction methods on logical state encoding and quantum error correction for the [7,1,3] Calderbank-Shor-Steane quantum error correction code in a nonequiprobable error environment. We determine the optimum number of verification steps to be used in Shor state construction and whether Shor states without verification are usable for practical quantum computation. These results are compared to the same processes of encoding and error correction where Shor states are not used. We demonstrate that the construction of logical zero states with no first order error terms may not require the complete edifice of quantum fault tolerance. With respect to error correction, we show for a particular initial state that error correction using a single qubit for syndrome measurement yields a similar output state accuracy to error correction using Shor states as syndrome qubits. In addition, we demonstrate that error correction with Shor states has an inherent sensitivity to bit-flip errors.

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I. INTRODUCTION

Quantum fault tolerance [1–4] is the framework which allows for accurate implementation of quantum algorithms despite the inevitability of errors during the computation. This is done by assuring that an error that occurs on one qubit cannot spread to multiple qubits. Application of quantum error correction (QEC) then corrects the single qubit error [5–7].

However, utilizing the entirety of the fault tolerant framework promises to be an expensive proposition in terms of the number of qubits and implemented gates. Thus, it is worth exploring whether it is possible to relax some of the strict rules required by the framework. One way to do this may be by easing the construction requirements or simply not using Shor states as syndrome qubits when encoding logical computational states and applying error correction. In this paper we study the utilization of Shor states in the encoding of logical zero states and the application of error correction for the [7,1,3] Steane code [8] with the goal of limiting the number of required qubits and implemented gates.

A fault tolerant method for encoding a logical computational state in the Steane code is to apply fault tolerant error correction to any initial state of 7 qubits. This requires construction of proper ancilla syndrome qubits such that each ancilla interacts with no more than one of the 7 data qubits. For the Steane code appropriate syndrome qubits are four-qubit Shor states [2]. Shor states are simply Greenberger-Horne-Zeilinger (GHZ) states with Hadamard gates applied to each qubit. However, as the Shor states themselves are constructed in a noisy environment (here the nonequiprobable error environment), verification via parity checks is necessary to ensure accurate construction. Thus, in this paper, we first attempt to determine the number of Shor state verifications necessary to construct logical zero states or apply error correction with as high a fidelity as possible. We then ask whether using Shor states with fewer verification steps (thus using fewer ancilla qubits and requiring fewer gates) will provide sufficient accuracy to be used in the construction of logical zero states or the application of error correction. Finally, we explore whether Shor states are necessary at all in the construction of logical zeros and the application of error correction, or whether sufficient accuracy may be obtained using single qubits for syndrome measurement.

The error model used in this paper is a nonequiprobable Pauli operator error model [9]. As in [10], this model is a stochastic version of a biased noise model that can be formulated in terms of Hamiltonians coupling the system to an environment. In the model used here, however, the probabilities with which the different error types take place is left arbitrary: the environment causes qubits to undergo a σ_x^j error with probability p_x, a σ_y^j error with probability p_y, and a σ_z^j error with probability p_z, where σ_i^j, $i = x, y, z$ are the Pauli spin operators on qubit j. We assume that only qubits taking part in a gate operation will be subject to error and the error is modeled to occur after (perfect) gate implementation. Qubits not involved in a gate are assumed to be perfectly stored. While this represents an idealization, it is justifiable as all accuracy measures are calculated only to second order in the error probabilities p_i. In addition, non-equiprobable errors occur in the preparation and measurement of all qubits.

This paper builds on the previous work of Ref. [11] (see also [12]) in which the fault tolerant method of encoding logical zero states for the [7,1,3] code was compared to the gate sequence method of encoding to see which method led to more accurately encoded zero states. Though the gate sequence method is not fault tolerant (errors can propagate to multiple data qubits) it was found that the fidelity of the logical zero states constructed in this way is comparable to the fidelity of the states constructed using the fault tolerant method. Applying perfect error correction then revealed that the error probabilities were reduced to at least second order for both methods (third order for the fault tolerant method), implying the correctability of the errors and suggesting that either

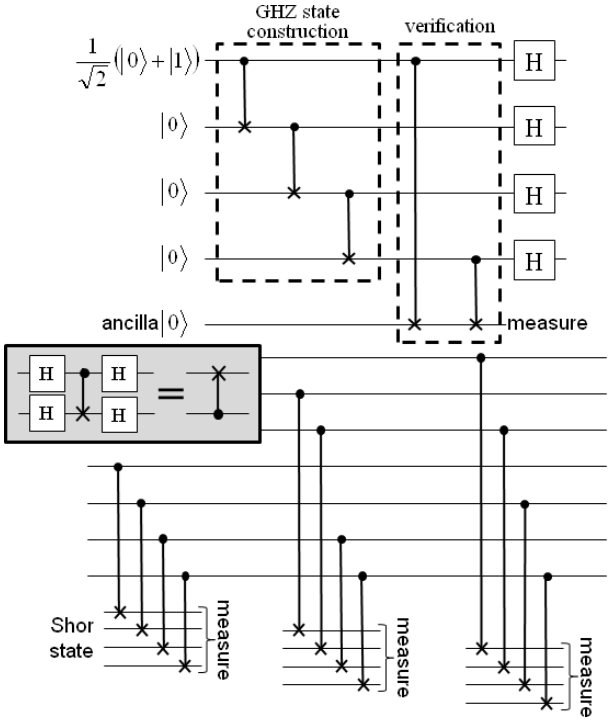

FIG. 1: Top: construction of a 4 qubit Shor state. CNOT gates are represented by (•) on the control qubit and (×) on the target qubit connected by a vertical line. H represents a Hadamard gate. The procedure entails constructing a GHZ state which is verified using ancilla qubits. Hadamard gates are applied to each qubit to complete Shor state construction. Bottom: syndrome measurements for the $[7, 1, 3]$ code. For fault tolerant syndrome measurement each Shor state ancilla qubit must interact with only one data qubit. The error syndrome is determined from the parity of the measurement outcomes of the Shor state ancilla qubits. To achieve fault tolerance each of the syndrome measurements is repeated twice. Box: a useful equality which allows us to avoid implementing Hadamard gates by reversing the control and target of CNOT gates. In our context, the CNOTs are reversed such that the ancilla qubits become the control and the data qubits become the target, as explained in the text.

method can be used for practical quantum computation. Here we work within the fault tolerant method in attempt to determine how best to construct Shor states for encoding and error correction. In both papers, however, a major goal is to determine whether accurate enough protocols can be implemented without invoking the full framework of quantum fault tolerance.

II. CONSTRUCTING SHOR STATES

A construction method for the four-qubit Shor states needed for the [7,1,3] QEC code is shown in Fig. 1. If the construction was done without error, no verification steps would be needed and the Shor state (without the final Hadamard gates as explained below) would be given by: $|\psi_{Shor}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. However, actual implementations of quantum computation will be done in a noisy environment and thus verifications may be useful. We simulate construction of Shor states in the nonequiprobable error environment including initialization and measurement errors with different verification strategies. We then determine which of the strategies produce the highest quality Shor states based on the fidelity of the constructed Shor states, the fidelity of logical zero states encoded fault tolerantly with the different Shor states used as syndrome qubits, and the fidelity of a state after noisy error correction when the different Shor states are used as syndrome qubits. The different strategies we use are: no verifcation steps, one verification step, and different possible two verification steps. The tenets of fault tolerance require that at least one verification step be applied so as to lower the probability of error to second order.

To construct the Shor state we start with four qubits that we attempt to initialize to the state zero. However, in this work, we assume that initialization itself is a noisy process subject to the same error model as qubits involved in a gate. Thus, the actual state of each initialized qubit is $\rho_i = (1-p_x-p_y)|0\rangle\langle 0| + (p_x+p_y)|1\rangle\langle 1|$. We then apply a Hadamard gate, H, to the first qubit. The nonequiprobable error environment causes imperfections in the gate such that the actual evolution of an attempted Hadamard on a single qubit j in the state ρ is:

$$\sum_{a}^{0,x,y,z} p_a \sigma_a^j H_j \rho H_j^\dagger \sigma_a^j, \tag{1}$$

where σ_0^j is the identity matrix, $p_0 = 1 - \sum_{\ell=x,y,z} p_\ell$, and the terms $K_a^j = \sqrt{p_a}\sigma_a^j H_j$ can be regarded as Kraus operators for the Hadamard evolution. The Hadamard is followed by a series of CNOT gates. The attempted performance of the CNOT gate with control qubit j and target qubit k, $\mathrm{C}_j\mathrm{NOT}_k$, in the nonequiprobable error environment on any state ρ actually implements:

$$\sum_{a,b}^{0,x,y,z} p_a p_b \sigma_a^j \sigma_b^k \mathrm{C}_j\mathrm{NOT}_k \rho \mathrm{C}_j\mathrm{NOT}_k^\dagger \sigma_a^j \sigma_b^k, \tag{2}$$

where terms $A_{a,b}^{j,k} = \sqrt{p_a p_b}\sigma_a^j \sigma_b^k \mathrm{C}_j\mathrm{NOT}_k$ can be regarded as the 16 Kraus operators. Shor state construction requires three CNOT gates, shown in Fig. 1, and thus the final Shor state is given by

$$\begin{aligned}\rho_{Shor-err} &= \sum_{a,b,c,d,e,f,g}^{0,x,y,z} A_{f,g}^{3,4} A_{d,e}^{2,3} A_{b,c}^{1,2} K_a^1 \rho_i^{\otimes 4} \\ &\times (K_a^1)^\dagger (A_{b,c}^{1,2})^\dagger (A_{d,e}^{2,3})^\dagger (A_{f,g}^{3,4})^\dagger. \end{aligned}\tag{3}$$

Simply applying the above described gate sequence does not build Shor states in a fault tolerant fashion. This is because multiple errors in Shor state construction can propagate into the data qubits when the Shor

TABLE I: Relevant fidelity measures for Shor states and encoded logical zeros from different construction methods: Shor state without verification, Shor state with one verification, Shor state with two verifications, and the accuracy of logical zero construction using single-qubit ancilla for syndrome measurements instead of Shor states. The accuracy measures are the fidelity of the Shor state itself, the fidelity of the seven physical qubits making up the logical zero state, the fidelity of the one qubit of information stored in the seven physical qubits, and the fidelity after perfect error corrction has been applied to the constructed encoded zero states.

	no verifications	1 verification	2 verifications	1-Qubit ancilla
Shor fidelity	$1-10p_x-11p_y-7p_z$	$1-5p_x-6p_y-10p_z$	$1-5p_x-6p_y-13p_z$	
7-Qubit fidelity	$1-85p_x-37p_y-12p_z$	$1-55p_x-19p_y-12p_z$	$1-55p_x-19p_y-12p_z$	$1-49p_x-19p_y-12p_z$
1-Qubit fidelity	$1-25p_x-11p_y$	$1-19p_x-7p_y$	$1-19p_x-7p_y$	$1-15p_x-7p_y$
after QEC	$1-92p_x^2-74p_xp_y-14p_y^2$	1	1	$1-26p_x^2-6p_xp_y$

states are used for syndrome measurement. We need to test the Shor states to ensure that multiple errors have not taken place. This is done utilizing an ancilla qubit, initially in the state $|0\rangle$, adjoined to the Shor state to measure the parity of random pairs of qubits [2]. Should the test fail (the ancilla qubit measurement yields a $|1\rangle$), the Shor state is immediately discarded. Of course, the ancilla qubit initialization and the CNOT gate implementations for this parity check are themselves performed in the nonequiprobable error environment and thus follow the dynamics described above. We utilize an initial ancilla qubit to measure the parity of qubits 1 and 4. Applying additional verification steps using additional ancilla may, if the CNOTs themselves are not too error prone, further ensure the lack of errors in the constructed Shor states. A second ancilla can recheck the parity of the qubits checked with the previous ancilla, or check the parity between other Shor state qubits. We have simulated every possible combination for the second parity measurement and this choice has little effect on any of our accuracy measures.

Our first accuracy measure for the Shor states constructed with different numbers of verifications is the fidelity of the constructed Shor state as compared to a perfect Shor state, $F = \langle\psi_{Shor}|\rho_{Shor-err}|\psi_{Shor}\rangle$. The fidelity results for Shor states with zero, one, and two parity verifications are shown to first order in error probability in the first row of Table I. Note that to first order the fidelity for Shor states of two verification is independent of which qubits are used for the second verification. The second order error probability terms, however, will depend on the choice of qubits for the second verification.

Comparing the fidelity of the three Shor states we see that the Shor state with only one verification leads to a higher fidelity than the Shor states with zero or two verifications. This apparently demonstrates the utility of performing a verification step on the Shor states in order to suppress errors that occur during the Shor state construction. However, applying a second verification step does not give enough of a benefit to outweigh additional errors that may occur during the verification procedure itself.

III. ENCODING WITH SHOR STATES

The fidelity reported above is a good measure of accuracy for the Shor state in and of itself. However, our purpose for constructing Shor states is to use them to encode logical zero states and implement fault tolerant error correction. It is possible that different errors in the Shor state construction will have more or less of an effect on the accuracy with which these protocols can be performed. Thus, another way to quantify the quality of the Shor states is to simulate their utilization in the encoding of logical zero states and in the performance of error correction and report on the accuracy with which these protocols are implemented.

We first turn to the construction of logical zero states. To do this in a fault tolerant manner we start with 7 qubits all noisily initialized to the state zero. Though this initialization is not perfect we choose to not perform the first set (bit-flip) of syndrome measurements as their utility in correcting an initialization error is outweighed by the noise inherent in applying the necessary syndrome measurments. Instead, we immediately measure the three phase flip syndromes (each one of the three twice) with Shor states as the syndrome qubits. We can measure the syndrome without Hadamard gates if we reverse the roles of the control and target qubits for the CNOT gates, and measure the Shor state qubits (noisily) in the x-basis, as explained in [11]. In this paper, we analyze the scenario where all four qubits are measured as zero.

Attempting this construction in the nonequiprobable error environment using Shor states with different numbers of applied verifications will result in logical zero states with different degrees of accuracy. We can measure this accuracy in a number of ways. The first way is simply to look at the fidelity of the seven qubit logical zero state. The accuracy of this state gives an idea as to how well the entire encoding process was performed. Alternatively, one may look at the fidelity of only the one qubit of encoded information. This is the only qubit of information that is actually of importance and, if it is protected, the state of the rest of the system is irrelevant. Measuring the fidelity of this one logical qubit is done by (noiselessly) decoding the constructed logical zero state, tracing out all qubits but the first, and comparing the state of the remaining qubit with the zero state on a

single qubit. Both of these fidelity measures have been calculated for logical zero states constructed using Shor states of zero, one, and two verification parity checks, and are given in Table I.

Errors affecting the logical zero state may also be of varying degrees of severity. Applying perfect error correction allows us to test the 'correctability' of the types of errors that occur during the encoding. If even perfect error correction cannot (to first order) correct the errors in the logical zero state then the encoding method cannot be used for practical implementations of quantum computation. We apply perfect error correction to the states constructed using the three types of Shor states (with different numbers of verifications) and calculate the fidelity measure of the output state. These fidelities are given in Table I, and corroborate our previous observations that applying one verifiction to Shor states is optimal. Applying no verification steps to the Shor states leads to lower fidelities for the logical zero states, and applying two verifications does not raise the fidelity. Perfect error correction applied to logical zero states encoded using Shor states with one or two verifications gives unit fidelity up to third order. However, perfect error correction applied to logical zero states encoded using Shor states with no verifications, suppresses errors to second order implying that these states may also be useable for practical quantum computation.

We compare the above cases of Shor state syndrome measurement with a logical zero encoding method in which a single ancilla qubit is used for each syndrome measurement. This method does not meet the standards of fault tolerance since an error on the single ancilla qubit, be it an initialization error or an error in one of the syndrome measurement CNOT gates, can spread to multiple data qubits. However, using one ancilla qubit removes the need to construct Shor states thus lowering the number of gates to be performed. The logical zero fidelity measures defined above are calculated for the single qubit syndrome measurement construction method and are shown in Table I. Comparing these fidelity measures to those calculated for Shor state based encoding, we find that using single qubit ancilla leads to higher fidelity logical zero states. However, upon application of perfect error correction the error probabilities are suppressed only to second order, unlike the logical zero states constructed using Shor states for which the second order error probability terms are also suppressed.

IV. QUANTUM ERROR CORRECTION WITH SHOR STATES

We now consider the accuracy with which the different Shor states can be used as syndrome ancilla qubits for quantum error correction. The arbitrary single-qubit initial state we would like to protect is assumed to have been perfectly encoded via the $[7,1,3]$ gate encoding sequence: $|\psi\rangle = \cos\alpha|0_L\rangle + e^{i\beta}\sin\alpha|1_L\rangle$, where $|0_L\rangle$ and $|1_L\rangle$ represent the seven qubit logical $|0\rangle$ and $|1\rangle$ states respectively. We assume the environment possibly causes an error such that, before error correction, the system is in a mixed state of no error and all possible single qubit errors:

$$\rho_{err} = (1-7(p_x+p_y+p_z))|\psi\rangle\langle\psi| + \sum_{i=1}^{7}\sum_{a}^{x,y,z} p_a \sigma_a^i |\psi\rangle\langle\psi| {\sigma_a^i}^\dagger. \tag{4}$$

Because there are only single qubit errors in the system state, the error can be corrected by perfect application of the [7,1,3] code.

To perform error correction in a fault tolerant manner, Shor states with at least one verification must be used for syndrome measurements. We apply error correction to the state ρ_{err} in the nonequiprobable error environment by implementing the three bit-flip syndrome measurements followed by three phase-flip syndrome measurements using Shor states with different numbers of verifications as the syndrome qubits. Each syndrome measurement is repeated twice to account for errors that may have occurred during the syndrome measurement itself. We again only analyze the scenario where all four ancilla qubits are measured as zero. We quantify the quality of the error correction via fidelity measures comparing the final state after error correction to the pre-encoded arbitrary state. The fidelities of the seven qubits and one logical qubit state are given, to first order, in Table II.

Comparing the seven-qubit fidelities of the QEC procedure utilizing Shor states with different numbers of verifications, we first note that the fidelities for Shor states with one and two verifications are identical up to second order terms. This fidelity is higher than that attained by performing QEC using a Shor state with no verifications, again confirming that while performing verification of the Shor state is important, there is no benefit gained from performing a second verification step. The fidelities exhibit little dependence on the initial state of the qubit, α and β only appear in second order terms of the zero verification Shor state QEC procedure fidelity. Furthermore, regardless of the Shor state used, the p_x error is dominant implying that bit-flips are more harmful to the error correction procedure than phase flips.

Similar trends hold when comparing the single qubit fidelities except that all of the single qubit fidelities depend strongly on the initial state. These fidelities are highest when $\alpha = 0, \frac{\pi}{2}$, at which point the first and second order p_z terms drop from the fidelity expression. The fidelities are lowest when $\alpha = \frac{\pi}{4}$. Once again p_x is the dominant error term. We note that the presence of first order terms in the fidelity measures indicate that, in this case, noisy QEC cannot output a state with no first-order error probability terms. Practical quantum error correction in this case is thus reduced to minimizing the coefficients of these first order terms.

We compare the above QEC performance with that of error correction done without Shor states, instead using a single (noisily initialized) ancilla qubit for syndrome mea-

TABLE II: Fidelity measures for error correction applied to the state ρ_{err} utilizing Shor states with different numbers of verifications or a single ancilla qubit for syndrome measurement. In the Table $a = \cos[4\alpha]$ and $b = \cos[2\beta]\sin[2\alpha]^2$. In this case the bit flip syndrome measurements were done first.

	no verifications	1 verification	2 verifications	1-Qubit ancilla
7-Qubit fidelity	$1-85p_x-25p_y-7p_z$	$1-55p_x-7p_y-7p_z$	$1-55p_x-7p_y-7p_z$	$1-49p_x-7p_y-7p_z$
1-Qubit fidelity	$1-(\frac{81}{4}+\frac{27}{4}a-\frac{27}{2}b)p_x$ $-(\frac{25}{4}+\frac{3}{4}a-\frac{5}{2}b)p_y$ $-\frac{3}{2}(1-a)p_z$	$1-(\frac{57}{4}+\frac{19}{4}a-\frac{19}{2}b)p_x$ $-(\frac{13}{4}-\frac{1}{4}a-\frac{1}{2}b)p_y$ $-\frac{3}{2}(1-a)p_z$	$1-(\frac{57}{4}+\frac{19}{4}a-\frac{19}{2}b)p_x$ $-(\frac{13}{4}-\frac{1}{4}a-\frac{1}{2}b)p_y$ $-\frac{3}{2}(1-a)p_z$	$1-(\frac{45}{4}+\frac{15}{4}a-\frac{15}{2}b)p_x$ $-(\frac{13}{4}-\frac{1}{4}a-\frac{1}{2}b)p_y$ $-\frac{3}{2}(1-a)p_z$

surement. While this scheme certainly does not meet the criteria for fault tolerance, it does allow us to implement QEC with fewer qubits, and the lack of possible error from the construction of the Shor states may, and in fact does, yield an improved resulting fidelity. The fidelities for this case are shown in the last column of Table II.

A. Why Bit Flips?

In all of the above, σ_x errors dominate the loss of fidelity. There are a number of possible reasons as to why this may be so. The first is because the bit flip syndrome measurements were implemented first, and thus σ_x errors that may occur during phase flip syndrome measurements are not corrected. A second possibility is that the use of (noisy) Shor states may cause the effect of σ_x errors to be more pronounced. In this section we clarify this issue by carrying out a series of simulations designed to isolate the cause of increased sensitivity to σ_x errors.

Our first step is to repeat the above error correction calculations implementing the phase flip syndrome measurements first. The fidelities of the resulting states are shown in Table III. Let us first compare the cases where the syndrome measurement was done with a single ancilla qubit. In this case the coefficients of the p_x and p_z terms in the seven qubit fidelity simply switch places while the p_y coefficient remains constant. Similarly in the one-qubit fidelity the p_y coefficient remains constant while the values of the p_x and p_z terms approximately trade values (modulo the contribution of the initial state). This alone suggests that the dominance of the p_x term in the original simulations was simply because the bit-flip syndrome measurements were done first. When the phase-flip syndrome measurements are done first p_z replaces p_x. However, when looking at the QEC simulations that utilize Shor states for syndrome measurements we do not find the same trade-off. Instead, though the p_z error coefficients grow and (in most cases) becomes dominant, we find much less of a reduction of the p_x error coefficients. This suggests that there is something inherent in the use of the (noisy) Shor states that leads to this type of error.

To further explore this point we perform two additional sets of QEC simulations. In the first simulations we utilize perfect Shor states but allow errors (due to the nonequiprobable error environment) in the error correction process (including syndrome measurement). In the second we use Shor states constructed in the nonequiprobable error environment (with one verification) but the error correction itself (including syndrome measurements) is perfect. Both are done with bit-flip syndrome measurements first and with phase-flip syndrome measurements first. When perfect Shor states are used, but the error correction is noisy, we find that the dominant error depends on which set of syndrome measurements is done first, if phase correction is done first σ_z errors dominate and vice-versa. The other error type is significantly diminished. When noisy Shor states are used with perfectly implemented error correction we find that which syndrome is done first makes little difference: σ_x errors dominate and the fidelities do not contain a first order term for σ_z errors. The various fidelity measures are displayed in Table IV.

Taken together these simulations imply that when noisy Shor states are utilized for syndrome measurement in the Steane code there is a significant bias towards bit-flip errors. A possible solution is to concatonate into a three-qubit bit-flip QEC code for another level of error correction. This could significantly reduce the sensitivity to bit-flip errors without the resource cost of concatenation into another level of the seven-qubit Steane code.

V. CONCLUSION

In conclusion, we have calculated quality metrics for different Shor states used as syndrome measurement ancilla qubits for the [7,1,3] CSS QEC code operating in a nonequiprobable error environment. The results suggest that while a Shor state constructed in this error environment with one parity check verification is optimal for suppressing errors in the construction of logical zero states, Shor states with no checks will also suppress error probability terms in the fidelity to second order. In addition, error correction applied without Shor states, instead using single qubit ancilla for syndrome measurement, leads to logical zero states with higher fidelity but errors that are less correctable as identified by fidelity after perfect error correction.

For error correction applied in a nonequiprobable error environment using the seven qubit Steane code, our simulations show that not using Shor states leads to a corrected state with higher fidelity than using Shor states. In addition, we noted that bit-flip errors are dominant whether Shor states are used or not. We first suggested that this was due to the fact that the bit-flip syndrome measurements were done first, meaning that uncorrected bit-flips may accumulate during phase-flip

TABLE III: Fidelity measures for error correction applied to the state ρ_{err} utilizing Shor states with different numbers of verifications or a single ancilla qubit for syndrome measurement. In the Table $a = \cos[4\alpha]$ and $b = \cos[2\beta]\sin[2\alpha]^2$. In this case the phase flip syndrome measurements were done first.

	no verifications	1 verification	2 verifications	1-Qubit ancilla
7-Qubit fidelity	$1 - 61p_x - 25p_y - 55p_z$	$1 - 31p_x - 7p_y - 55p_z$	$1 - 31p_x - 7p_y - 55p_z$	$1 - 7p_x - 7p_y - 49p_z$
1-Qubit fidelity	$1 - (\frac{61}{4} - \frac{49}{4}a - \frac{3}{2}b)p_x$ $-(\frac{33}{4} - \frac{21}{4}a - \frac{1}{2}b)p_y$ $-\frac{27}{2}(1-a)p_z$	$1 - (\frac{33}{4} - \frac{21}{4}a - \frac{21}{2}b)p_x$ $-(\frac{13}{4} - \frac{1}{4}a - \frac{1}{2}b)p_y$ $-\frac{27}{2}(1-a)p_z$	$1 - (\frac{33}{4} - \frac{21}{4}a - \frac{3}{2}b)p_x$ $-(\frac{13}{4} - \frac{1}{4}a - \frac{1}{2}b)p_y$ $-\frac{27}{2}(1-a)p_z$	$1 - (\frac{9}{4} + \frac{3}{4}a - \frac{3}{2}b)p_x$ $(\frac{13}{4} - \frac{1}{4}a - \frac{1}{2}b)p_y$ $-\frac{25}{2}(1-a)p_z$

TABLE IV: Fidelity measures for quantum error correction applied to the state ρ_{err} with perfect Shor states and noisy error correction, and noisy Shor states with perfect error correction. Both cases were done with the σ_x syndrome measurements first and the σ_z syndrome measurements first. In the Table $a = \cos[4\alpha]$ and $c = \cos[2\beta]$.

		bit-flip first	phase-flip first
Noisy QEC Perfect Shor States	7-Qubit fidelity	$1 - 31p_x - 7p_y - 7p_z$	$1 - 7p_x - 7p_y - 55p_z$
	1-Qubit fidelity	$1 - (\frac{33}{4} + \frac{11}{4}(a - c + ac)p_x$ $-(\frac{13}{4} - \frac{1}{4}(a + c - ac)p_y$ $-\frac{3}{2}(1-a)p_z$	$1 - (\frac{9}{4} + \frac{3}{4}a - \frac{3}{2}b)p_x$ $-(\frac{13}{4} - \frac{1}{4}a - \frac{1}{2}b)p_y$ $-\frac{27}{2}(1-a)p_z$
Perfect QEC Noisy Shor States	7-Qubit fidelity	$1 - 24p_x$	$1 - 24p_x$
	1-Qubit fidelity	$1 - (6 + 2a - 4b)p_x$	$1 - (6 + 6a)p_x$

syndrome measurements. Simulations switching the order of the syndrome measurements demonstrated that this is correct when using single qubit ancillae for syndrome measurement, but does not completely explain the results of simulations using Shor states. Further simulations indicated an inherent sensitivity towards bit-flip errors when Shor states are used. We suggested that this could be overcome by concatenating with a three-qubit QEC code that protects against bit-flip errors.

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