# Measuring Non-Zero Syndromes

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Quantum error correction codes are designed to identify and locate errors such that they can be properly corrected by recovery operations. Here, we demonstrate that, in realistic error environments, if a syndrome measurement indicates an error has occurred the assumed recovery will not, in general, lead to a corrected state. The efficacy of the recovery operation properly depends on the likelihood of different error types, the relative probability an error may have occurred at strategic points during the error correction process itself, and on the error correlations between the qubits taking part in a two qubit gate.

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Quantum error correction (QEC) [1–3] will be a vital component of any realistic realization of quantum computation. To implement QEC a given number of qubits of quantum information are stored in a larger number of physical qubits. Should an error occur on a physical qubit, measurement of the parity between groups of physical qubits, known as a syndrome measurement, will allow us to identify, locate, and thus correct, the error.

QEC alone, however, is not sufficient to guarantee successful quantum computation. One of the primary reasons for this is because the encoding of the quantum information into the QEC code and the implementation of the syndrome measurements are themselves imperfect. The framework that allows for successful quantum computation despite errors in basic computational components is quantum fault tolerance (QFT) [4–7]. Computing within the QFT framework requires: implementing quantum operations in such a way that errors will not spread to multiple qubits, and the repetition of certain protocols in order to relegate errors to second order in error probability. For example, QFT dictates repeating each syndrome measurement until the same outcome is measured twice in a row. In this way an error during syndrome measurement must occur twice for the syndrome measurement to be incorrect.

In this work we analyze what happens when a non-zero syndrome is measured signaling the presence of an error. We show that the standard and expected response, applying the appropriate recovery operation will not, in general, correct the state of the system. Whether it will or not depends on a number of factors including: the type of errors that are likely to occur, the ratio of the probability of an error on the qubit identified by the syndrome readout to the probability of an error occurring during the syndrome measurement itself, and the correlation of errors between qubits under the evolution of a two-qubit gate. With respect to the first factor, dominant $\sigma_x$ errors in Shor state construction lead to high fidelity. With respect to the second factor, the larger the ratio the better the recovery operation will fix the error. With respect to the third factor, the larger the correlation the more accurate the identification of the qubit which has experienced an error. To demonstrate how and why this is so we perform simulations using the [7,1,3] or Steane QEC code [8].

We start with an arbitrary pure state $|\psi(\alpha,\beta)\rangle = \cos\alpha|0\rangle + e^{-i\beta}\sin\alpha|1\rangle$ perfectly encoded into seven qubits of [7,1,3] QEC code except for one error. For the moment it is not important which error occurs just that the error certainly happened and it has affected only one of the seven 'data' qubits. Certainly, perfectly implemented QEC will detect the error and the appropriate recovery operation (perfectly applied) will correct the error. If QEC is noisy, assuming it is not too noisy, we would still expect it to work, especially if its implementation is loyal to the tenets of QFT. Assuming instead a probabilistic error on a data qubit, with small probability, $p \ll 1$, the likelihood is that syndrome measurements will not detect the error and the data qubits will be projected back into their proper state. In realistic systems, where the syndrome measurement itself must be assumed imperfect, measuring each syndrome twice will ensure that the measurement outcome did not arise from an error during the syndrome measurement. Thus, if a non-zero syndrome is measured (twice) we would assume that the error arose from the data qubit, apply the appropriate recovery operation, and recover the correct state. However, this assumption is incorrect. In this paper we demonstrate that, in general, the recovery operation will not correct the state of the system.

To explore the consequences of measuring a non-zero syndrome we perform simulations of the [7,1,3] QEC code in a nonequiprobable Pauli operator error environment [9]. As in [10], this model is a stochastic version of a biased noise model that can be formulated in terms of Hamiltonians coupling the system to an environment. In the model used here the probabilities with which the different error types take place is left arbitrary: qubits undergo a $\sigma_x^j$ error with probability $p_x$, a $\sigma_y^j$ error with probability $p_y$, and a $\sigma_z^j$ error with probability $p_z$, where $\sigma_i^j$, $i = x, y, z$ are the Pauli spin operators on qubit $j$. We assume that qubits taking part in a gate operation, initialization or measurement will be subject to error and, for now, we assume that errors are completely uncorrelated. Qubits not involved in a gate are assumed to be perfectly stored. This idealization is partially justified in that it is generally assumed that idle qubits are less likely to undergo error than those involved in gates (see for example [11]). In addition, in this paper, accuracy measures are calculated only to second order in the error probabilities $p_i$ thus the effect of ignoring storage errors is likely minimal.

Several methods of syndrome measurement for the [7,1,3] QEC code are compatible with QFT requirements [8, 12]. Here, we utilize four-qubit Shor states [5] for syndrome measurements. Shor states are simply GHZ states with a Hadamard applied to each qubit. Their construction must also be performed in the nonequiprobable error environment and, following the dictates of QFT, undergo one verification step [13]. The syndrome measurement then consists of controlled-NOT (CNOT) gates between the data qubits and Shor state qubits and the parity of the measurements of the four Shor state qubits gives the result of the syndrome measurement. Note that no Shor state qubit ever interacts with more than one data qubit thus stemming the spread of any possible error. As each syndrome measurement is done twice, to comply with the requirements of fault tolerance, 12 Shor states are needed for QEC application. The complete QEC process (but with each syndrome measurement pictured only once) is shown in Fig. 1.

As a baseline we simulate a state with a definite error undergoing noisy QEC. We will calculate the probability that the error will be detected and the fidelity of the state after error correction and recovery operation. Let $\rho(\alpha,\beta) = |\psi_L(\alpha,\beta)\rangle\langle\psi_L(\alpha,\beta)|$ where $|\psi_L(\alpha,\beta)\rangle = \cos\alpha|0_L\rangle + e^{-i\beta}\sin\alpha|1_L\rangle$ and $|0_L\rangle$, $|1_L\rangle$ are the logical basis state for the [7,1,3] code. Our initial state is then $\sigma_z^1\rho(\alpha,\beta)\sigma_z^1$. If QEC was done perfectly the error would be identified by the final phase-flip syndrome measurement (as shown in Fig. 1) with unit probability and would be corrected by the appropriate recovery operation, $\sigma_z^1$.

When QEC is performed in the nonequiprobable error environment (with each syndrome measurement implemented twice) the probability of detecting the error is no longer unity and is instead given by $1 - 167p_x - 257p_y - 209p_z$. Upon applying the appropriate recovery operation the fidelity, $F(p_x, p_y, p_z, \alpha, \beta) = \langle\psi_L(\alpha,\beta)|\rho_f|\psi_L(\alpha,\beta)\rangle$ of the output state, $\rho_f$, is $1 - 73p_x - 19p_y - 7p_z$. This result is about what would be expected given the gate imperfections. The imbalance between $p_x$ and $p_z$ is due to the $\sigma_x$ syndromes being measured first, and the use of imperfect Shor states [13].

We now turn to the case where $\sigma_z^1$ error is possible, and the initial state is thus $\rho_{p_z^1} = (1-p_z)\rho(\alpha,\beta) + p_z\sigma_z^1\rho(\alpha,\beta)\sigma_z^1$. The fidelity of the output state after noisy QEC reveals a $\sigma_z^1$ error and the appropriate (noisy) recovery operation is applied is shown in Fig. 2. Immediately we note the wide range of possible fidelity values based on the error probabilities. Even when all error probabilities are $\leq 10^{-5}$ the fidelities range from .5 to .9999. Specifically we note that when $p_x = p_y = p_z < 10^{-6}$ a weak depolarizing environment, the fidelity is .708. How could the fidelity values be so low even though proper QFT protocol was followed? To analyze this we look at a contrived QEC process in which only one gate is subject to errors.

Utilizing the same initial state, $\rho_{p_z^1}$, we apply perfect error correction (using perfect Shor states and repeat-

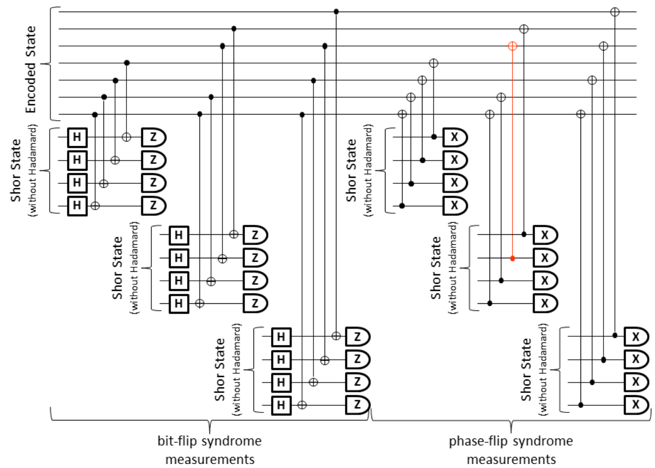


FIG. 1: Fault tolerant bit-flip and phase-flip syndrome measurements for the [7,1,3] code using Shor states (here the Shor states are assumed to have not had the Hadamard gates applied). CNOT gates are represented by (•) on the control qubit and (⊕) on the target qubit connected by a vertical line. $H$ represents a Hadamard gate. The error syndrome is determined from the parity of the measurement outcomes of the Shor state ancilla qubits. To achieve fault tolerance each of the syndrome measurements is repeated twice. In later simulations only the highlighted CNOT gate (between data qubit 3 and the second Shor state qubit in the second phase-flip syndrome) is subject to error and all other gates are simulated without error.

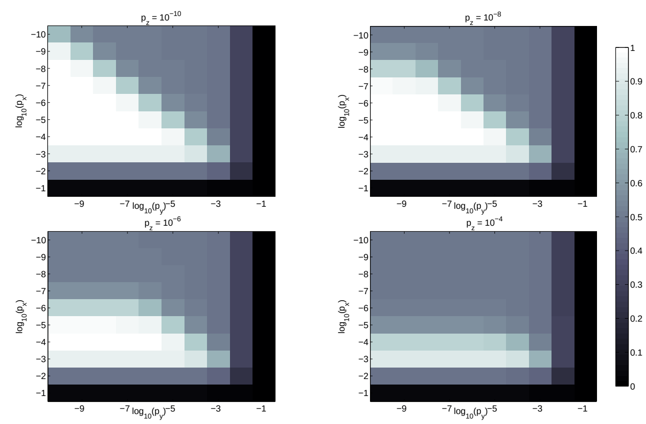

FIG. 2: Fidelity of output state after noisy error correction on the state $\rho_{p_z^1}$ reveals a $\sigma_z^1$ error and a recovery operation is applied. Each subfigure is for a given $\sigma_z$ error probability.

ing each syndrome measurement twice) except for one CNOT gate during which the control and target qubits each independently undergo a phase-flip error with probability $p_z$. We choose the one noisy CNOT gate to be the CNOT between the third data qubit and the second Shor state qubit during the second phase-flip syndrome measurement (the second time this syndrome is implemented only). Under this error model, if no error is detected (each syndrome measures zero twice in row), which will occur with probability $1-3p_z(1-p_z)$, the fidelity is given by $\frac{(1-p_z)(1-2p_z)}{1-3p_z+3p_z^2}$ which, in the limit $p_z \to 0$, goes to 1. However, if an error on the first qubit is detected (the final phase-flip syndrome measurement yields a one twice in a row), which will happen with probability $p_z(2-3p_z)$, the fidelity of the state after QEC is $\frac{1-2p_z}{2-3p_z}$ which, in the limit $p_z \to 0$ goes to .5. The reason why is clear. A measurement of 1 on the last phase-flip syndrome indicates a phase-flip error on either qubits 1, 3, 5 or 7. Were errors only to have occurred before QEC application then errors on qubit 3, 5, or 7 would have caused at least one additional syndrome measurement to be a one. However, errors during QEC application can allow an error on qubits 3, 5, or 7 to cause a measurement of one on only the final syndrome. Thus, in our simulation, there are two possible error sources that can induce a final phase-flip syndrome measurement of one: the error on qubit one of the initial state, and the error on qubit 3 during the only noisy syndrome measurement CNOT. This latter error happens during the QEC implementation itself and is thus not detected by any other syndrome measurement. As both error paths occur with probability $p_z$, the final state after error correction is a mixed state with each of these two possibilities having equal weight. The recovery operation will correct only one of the states of the mixture leading to a fidelity of .5. Note that the probability of measuring this syndrome is *first* order in $p_z$.

The above analysis explains the low fidelity points seen in Fig. 2. If $p_z$ or $p_y$ is greater than $p_x$ we are in a situation similar to our contrived model. An error during QEC can cause a syndrome measure of one in the final phase-flip syndrome without being detected by previous syndrome measurements thus reducing the fidelity to be of order .5. However, if $p_x$ is dominant we will see that errors in the Shor state construction have the most weight causing the syndrome to correctly detect the original error.

Our simplified simulation suggests three strategies to successfully avoid low fidelity output states when measuring a non-zero syndrome. The first strategy is to arrange that the probablility of error in the initial state is higher than the probability of error during the gates of the syndrome measurement. Using our example this would mean raising the error probability on qubit one of the initial state thus raising the probability of that part of the post-error correction mixed state and attaining a higher fidelity.

We analyze this strategy by simulating a perfect QEC sequence except for the same CNOT gate identified above. This time, however, our intial state will include an error on the first qubit larger than the error probability of the CNOT gate: $\rho_{kp_z^1} = (1-kp_z)\rho(\alpha,\beta) + kp_z\sigma_z^1\rho(\alpha,\beta)\sigma_z^1$ where $k$ is a positive, non-zero constant. If no error is detected, which will occur with probability $1-(2+k)p_z+3kp_z^2$, the fidelity of the state after QEC is, $\frac{(1-2p_z)(1-kp_z)}{1-(2+k)p_z+2kp_z^2}$ which, as above, will go 1 in the limit $p_z \to 0$. If the error on the first qubit is detected, which will occur with probability $p_z(1+k-3kp_z)$ the fidelity will depend on $k$ as follows: $\frac{k-2kp_z}{1+k-3kp_z}$, which in the small $p_z$ limit goes as $F(k) = \frac{k}{k+1}$. The top line of the bottom inset to Fig. 3 is a log-log plot of $1-F(k)$. Note that to reach a fidelity of $1-10^{-5}$ requires a $k$ of $10^5$.

During an actual computation ensuring that the initial state is more error prone then the gates of the QEC implementation can be realized by implementing many gates in a row before applying QEC. As explained in [14], if many gates are done without QEC the error probability on the data qubits will steadily increase. Nevertheless, all these errors are correctable via QEC applied after the sequence of gates. The increase in error probability on the data qubits becomes important if the syndrome measurement yields a one.

The second strategy that can be used to avoid low fidelity output states upon measuring a non-zero syndrome is to ensure correlated errors between the two qubits of a two-qubit gate. In our example we have assumed a possible $\sigma_z$ errors in the CNOT on qubit three and a Shor state qubit during the second phase-flip syndrome measurement. If the errors between qubit three and the Shor state qubit were correlated an error on qubit 3 could only occur if the same error happened on the Shor state qubit. The error on the Shor state qubit would be immediately detected by the second phase-flip syndrome measurement. Given that the second phase-flip syndrome measurement does not detect an error we can rest assured that there was no error on qubit 3. Then, a measurement of one on the final phase-flip syndrome certainly means that qubit one has an error.

To analyze this strategy we again simulate a perfect error correction sequence except for the same CNOT gate identified above. We now use a correlated error model for the CNOT such that a phase-flip error will occur independently on the control or target qubit with probability $p_z(1-c)$ and correlated phase-flip errors will occur on both the control and target qubits with probaility $cp_z$, where $c$ is a correlation parameter ranging between 0 and 1. Once again our intial state is $\rho_{p_z^1}$. If no error is detected, which will occur with probability $1+p_z(c-3+p_z(3-2c))$, the fidelity of the state after QEC is $\frac{(1-p_z)(1+p_z(c-2))}{1-p_z(3-3p_z+c(2p_z-1))}$ which in the limit

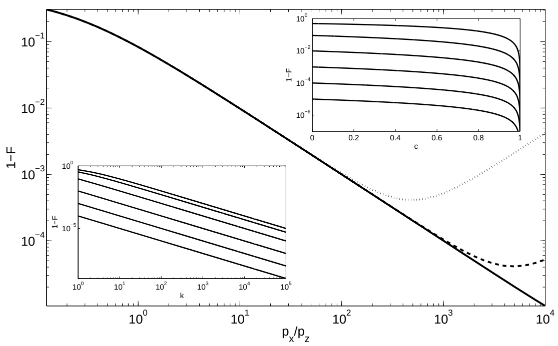


FIG. 3: One minus the fidelity versus $p_x/p_z$ for simulations that have one CNOT gate subject to $\sigma_z$ errors with probability $p_z$, and $\sigma_x$ errors with probability $p_x$ at all points of Shor state construction and verification. When $p_x$ is too high the fidelity again starts to decrease. Insets: Log-log plot of $1-F(c,k)$ as a function of initial state error probability strength $k$ (lower) for correlation $c = 0, .5, .9, .99, .999, .9999$ (top to bottom), and $c$ (upper) for initial state error probability strength $k = 1, 10, 100, 1000, 10000, 100000$ (top to bottom).

$p_z \to 0$ goes to 1 independent of $c$. If the error on the first qubit is detected, which will occur with probability $p_z(2-c) + p_z^2(-3+2c)$, the fidelity of the state after QEC and recovery will be $\frac{1-p_z(2-c)}{2-3p_z-c(1-2p_z)}$ which, in the limit $p_z \to 0$, is $F(c) = \frac{1}{2-c}$. $F(c)$ goes to .5 for $c = 0$ and 1 for $c = 1$. The top line of the top inset to Fig. 3 is a log-plot of $1-F(c)$ and demonstrates the importance of highly correlated errors if we hope to have a high fidelity output state from a QEC application that has a non-zero syndrome measurement. The possibility of engineering error correlations in a practical system would depend on the physical system.

When both strategies are used, we start with the initial state $\rho_{kp_z^1}$ and use a correlated error the identified CNOT gate, the probability of not detecting an error will be $1+p_z(c-2)+kp_z(p_z(3-2c)-1)$ in which case the fidelity of the state after QEC will be $\frac{1+p_z(c-2)(1-kp_z)}{1-(2-c+k)p_z+(2c-3)kp_z^2}$. The probability of detecting the error on the first qubit is $p_z(1-c+k+kp_z(2c-3))$ and the fidelity of the resulting state is then $\frac{k(1+p_z(c-2))}{1+k-3kp_z+c(2kp_z-1)}$ which, in the limit $p_z \to 0$, goes to:

$$F(c,k) = \frac{k}{1-c+k}. \quad (1)$$

$1-F(c,k)$ is plotted in the insets to Fig. 3 and will go to one in either the limit $c \to 1$ or $k \to \infty$.

The final strategy that can be used to avoid low fidelity output states upon measuring a non-zero syndrome is to utilize Shor states with dominant bit-flip errors. To demonstrate this we use our contrived error model where only one gate during QEC implementation is noisy, but now with Shor states that have been subjected to $\sigma_x$ errors during initialization of the qubits, construction gates, and verification gates with probability $p_x$. The fidelity of the output state then increases as the ratio of $p_x$ to the initial state and noisy CNOT error probability $p_z$ increases. This explains the success of the noisy QEC protocol when $p_x$ is the dominant error. The fidelity of the state after error correction as a function of the ratio $p_x/p_z$ is shown in Fig. 3.

In conclusion, we have demonstrated that, in realistic error environments, a non-zero syndrome measurement can cause output states to have unacceptably low fidelities. Whether this will occur depends on the relative probabilities of $\sigma_x$ errors in Shor state construction compared to $\sigma_y$ and $\sigma_z$ errors in the QEC implementation, the relative probability of errors on the input state compared to probability of error during syndrome measurement, and the strength of the error correlation between two qubit gates evolving under a two-qubit gate. Systems may be engineered so as to retain a high fidelity even upon a non-zero syndrome measurement. For example, implementing a sequence of gates with QEC applied only at the end will raise the probability of error on the input state above the error probability of the gates during QEC and thus increase the fidelity. While all of our simulations have been done in for the [7,1,3] QEC code the results are immediately applicable to other CSS codes and perhaps to a broader range of codes as well.

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